From Wednesday's In Class Work:

Let $P(x)$ be the power series

$$1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots = \sum_{k=0}^{\infty} \frac{(-x)^k}{k+1}$$

1. When $x = 1$, we have $P(1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$. AST $\Rightarrow$ $P(1)$ converges.
2. When $x = -1$, we have $P(-1) = \sum_{k=0}^{\infty} \frac{(1)^k}{k+1} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$.

   Harmonic series; diverges. (Integral Test would also work.)

   Notice: $\sum \frac{1}{k+1}$ diverging means that $\sum \frac{(-1)^k}{k+1}$ converges *conditionally*.
3. When $x = \frac{1}{2}$, we have $P(\frac{1}{2}) = \sum_{k=0}^{\infty} \frac{(-1/2)^k}{k+1} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2^k(k+1)}$.

   AST $\Rightarrow$ $P(\frac{1}{2})$ converges. $\sum_{k=0}^{\infty} \frac{1}{2^k(k+1)} \leq \sum_{k=0}^{\infty} \frac{1}{2^k} \Rightarrow P(\frac{1}{2})$ converges absolutely.

Recall – The Ratio Test:

Suppose that $\sum_{k=1}^{\infty} a_k$ is a series of non-zero terms, and that

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = L.$$

Then

1. If $L < 1$, then $\sum a_k$ converges absolutely.
2. If $L > 1$ (or if $L = \infty$), then $\sum a_k$ diverges.
3. If $L = 1$, the Ratio Test is inconclusive.

In Class Work

Find the interval of convergence for the following power series:

1. $\displaystyle\sum_{n=0}^{\infty}(n+1)(x-3)^n$
2. $\displaystyle\sum_{j=0}^{\infty}\frac{x^j}{j!}$

Solutions:

Find the interval of convergence for the following power series:

1. $\displaystyle\sum_{n=0}^{\infty}(n+1)(x-3)^n$
   - Using the ratio test, find the int. of conv., give or take endpoints:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{(n+2)|x-3|^{n+1}}{(n+1)|x-3|^n}\\ &= \lim_{n\to\infty}\frac{(n+2)|x-3|}{n+1}\\ &= |x-3|\lim_{n\to\infty}\frac{n+2}{n+1}\\ &= |x-3|\end{aligned}$$

$\Rightarrow \displaystyle\sum_{n=0}^{\infty}(n+1)(x-3)^n$ converges absolutely if $|x-3|<1$, diverges if $|x-3|>1$, i.e. converges absolutely when $-1<x-3<1$, or $2<x<4$, and diverges when $x<2$ or $x>4$.

Solutions:

1. (continued) $\sum_{n=0}^{\infty}(n+1)(x-3)^n$ converges absolutely when $2 < x < 4$, and diverges when $x < 2$ or $x > 4$.

   But what happens at $x = 2$ and $x = 4$?

   The ratio test is inconclusive, so check these cases individually.

   - When $x = 4$, the series we're dealing with is $\sum_{n=0}^{\infty}(n+1)(1)^n = \sum_{n=0}^{\infty}(n+1)$. This series obviously diverges.
   - When $x = 2$, the series we're dealing with is $\sum_{n=0}^{\infty}(-1)^n(n+1)$.

     Using the alternating series test, this series obviously diverges also.

   Therefore, the interval of convergence is $(2, 4)$.

Solutions:

Find the interval of convergence for the following power series:

2. $\displaystyle\sum_{j=0}^{\infty} \frac{x^j}{j!}$

- Using the ratio test, find the int. of conv., give or take endpoints:

$$\begin{aligned}\lim_{j\to\infty}\left|\frac{a_{j+1}}{a_j}\right| &= \lim_{j\to\infty}\frac{|x|^{j+1}j!}{|x|^j(j+1)!}\\ &= \lim_{j\to\infty}\frac{|x|}{j+1}\\ &= 0\end{aligned}$$

Since $\displaystyle\lim_{j\to\infty}\left|\frac{a_{j+1}}{a_j}\right| = 0 < 1$ **independent of x**, $\displaystyle\sum_{j=0}^{\infty}\frac{x^j}{j!}$ *always* converges absolutely, no matter what value of x you may choose to use.

Conclusion: The interval of convergence for $\displaystyle\sum_{k=0}^{\infty}\frac{x^k}{k!}$ is $(-\infty, \infty)$.