Recall: Integral Test for Positive-Term Series

Suppose that for all $x \geq m$, $a(x)$ is continuous, non-negative, and decreasing. Let $a_k = a(k)$ for all $k = m, m+1, \ldots$.

Then the series $\sum_{k=m}^{\infty} a_k$ and the integral $\int_m^{\infty} a(x)\,dx$ either both converge or both diverge.

Furthermore,

$$\int_m^{\infty} a(x)\,dx \leq \sum_{k=m}^{\infty} a_k \leq a_m + \int_m^{\infty} a(x)\,dx$$

which can be used to quickly find upper and lower bounds for a convergent sequence

Recall: Immediate Consequence of the Integral Test

Since $\dfrac{1}{x^p}$ is a continuous, positive, and decreasing function on $[a, \infty)$ for all $a > 0$ and for all $p > 0$, the integral test applies to $a(x) = \dfrac{1}{x^p}$ and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ for all $p > 0$.

Thus $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges whenever $\displaystyle\int_1^{\infty} \frac{1}{x^p}\, dx$ does, and diverges whenever the integral does.

So ...

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$$

Such series are called **p series**.

Recall: Remainders

- Partial Sums: $S_n = \sum_{k=1}^{n} a_k$
- $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n$
- For any n,

$$\begin{aligned} \sum_{k=1}^{\infty} a_k &= \sum_{k=1}^{n} a_k + \sum_{k=n+1}^{\infty} a_k \\ &= S_n + R_n \end{aligned}$$

- If the series converges to S, then $\lim_{n\to\infty} S_n = S$ *and* $\lim_{n\to\infty} R_n = 0$

In Class Work

Let $S = \sum_{k=1}^{\infty} \frac{k}{e^{k^2}}$.

1. Use the integral test to show this series converges (if you didn't Friday).

2. Use the integral test to find lower and upper limits for the value of S.

3. Find a number N such that the partial sum S_N approximates the sum of the series within 0.001.

Solutions:

1. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{e^{k^2}}$

   Check on a graph: the function is continuous, positive, and decreasing from about $x = 1$ on, so the integral test applies.

   Integral Test $\implies \displaystyle\sum_{k=1}^{\infty} \frac{k}{e^{k^2}}$ will do whatever $\displaystyle\int_1^{\infty} xe^{-x^2}\,dx$ does.

$$\int_1^{\infty} xe^{-x^2}\,dx = \int_{-1}^{-\infty} -\frac{1}{2}e^u\,du = -\frac{1}{2}\lim_{R\to\infty} e^u\Big|_{-1}^{-R} = \frac{1}{2e}.$$

   Since this integral converges, so does the series (although it does not converge to $\frac{1}{2e}$)

Solutions:

Let $S = \sum_{k=1}^{\infty} \frac{k}{e^{k^2}}$.

2. Use the integral test to find lower and upper limits for the value of S.

   Integral test $\Longrightarrow \int_1^\infty a(x)\,dx \le \sum_{k=1}^{\infty} a_k \le 1 + \int_0^\infty a(x)\,dx$.

   Since we found that $\int_1^\infty xe^{-x^2}\,dx = \frac{1}{2e}$,

$$\begin{aligned}
\int_1^\infty xe^{-x^2}\,dx \quad &\le S \le \quad a_1 + \int_1^\infty xe^{-x^2}\,dx \\
\frac{1}{2e} \quad &\le S \le \quad \frac{1}{e} + \frac{1}{2e} = \frac{3}{2e} \\
0.184 \quad &\le S \le \quad 0.552
\end{aligned}$$

Solutions:

3 Find a number N such that the partial sum S_N approximates the sum of the series within 0.001.

Need $R_N = \sum_{k=N+1}^{\infty} \leq 0.001$.

Since $R_N \leq \int_N^\infty a(x)\, dx$, suffices to find N so $\int_N^\infty a(x)\, dx \leq 0.001$.

$$\int_N^\infty xe^{-x^2}\, dx \leq 0.001 \Rightarrow -\frac{1}{2} \lim_{R\to\infty} e^{-x^2}\Big|_N^R \leq 0.001$$

$$\Rightarrow \frac{e^{-N^2}}{2} \leq 0.001 \Rightarrow e^{-N^2} \leq 0.002$$

$$\Rightarrow -N^2 \leq \ln(0.002) \Rightarrow N^2 \geq -\ln(0.002)$$

$$\Rightarrow N \geq \sqrt{-\ln(0.002)} = 2.49$$

Thus $\sum_{k=1}^{3} \frac{k}{e^{k^2}}$ is within 0.001 of $\sum_{k=1}^{\infty} \frac{k}{e^{k^2}}$.