Recall: Known Power Series

- $\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
- $\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
- $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$
- $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$

In Class Work

1. (a) Use a known power series to find a power series expansion for e^{-x^3}. What is the radius of convergence for this power series?

   (b) Find a power series expansion of $\int_0^1 e^{-x^3}\,dx$. Why are power series helpful here?

   (c) Approximate the value of this integral within 0.001.

2. *A Power Series for π:*

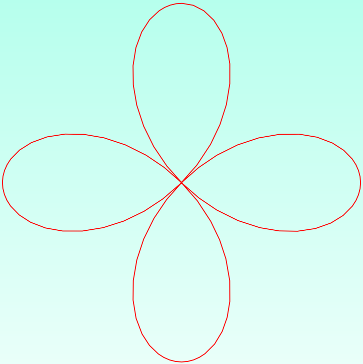
   (a) Find a power series expansion for $\frac{1}{1+x^2}$ based at $x=0$.

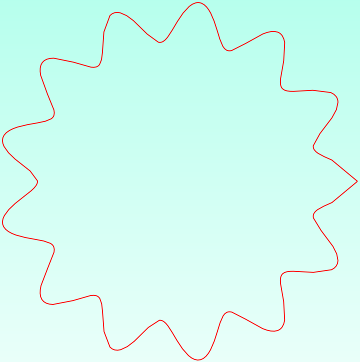
   (b) Find a power series expansion for $\arctan(x)$ based at $x=0$.

   (c) Find a power series expansion for $\frac{\pi}{4}=\arctan(1)$.

   (d) Find a power series expansion for π.

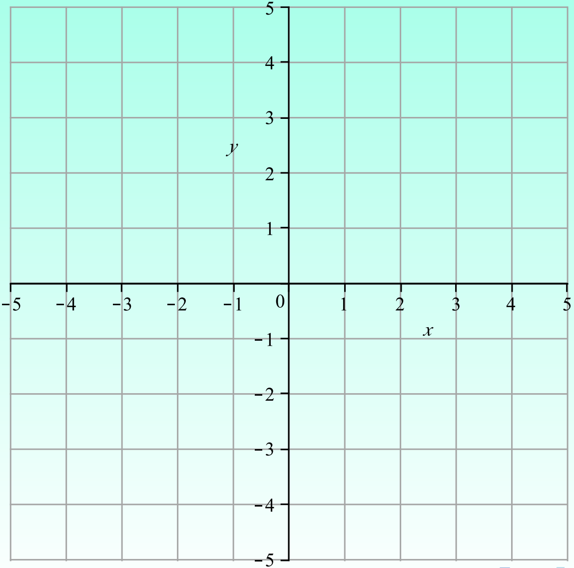
With polar coordinates, some new graphs are a snap!



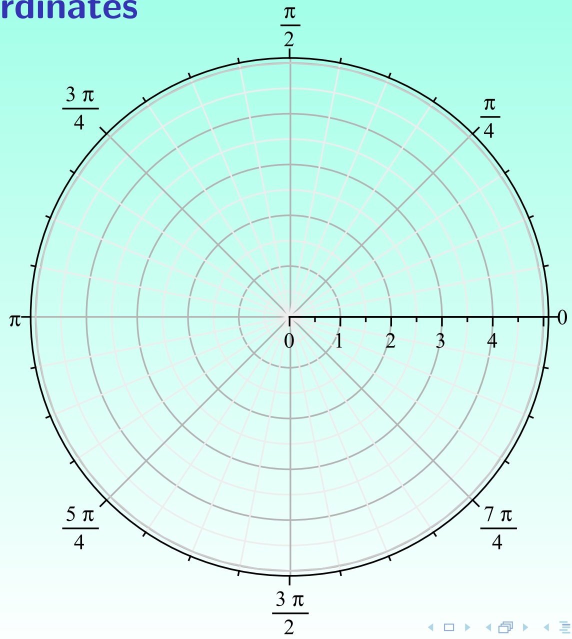


Each of these is just the graph of a single simple function.

Rectangular Coordinates



Polar Coordinates



Solutions:

1. Find a power series expansion of $\int_0^1 e^{-x^3}\,dx$. Approximate the value of this integral accurate within 0.001.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \Rightarrow \quad e^{-x^3} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{3k}}{k!}$$

$$\Rightarrow \quad \int_0^1 e^{-x^3}\,dx = \int_0^1 \left(\sum_{k=0}^{\infty} \frac{(-1)^k x^{3k}}{k!} \right) dx$$

$$\Rightarrow \quad \int_0^1 e^{-x^3}\,dx = \sum_0^{\infty} (-1)^k \left(\frac{x^{3k+1}}{(3k+1)k!} \right) \Bigg|_0^1$$

$$\Rightarrow \quad \int_0^1 e^{-x^3}\,dx = \sum_0^{\infty} (-1)^k \frac{1}{(3k+1)k!}$$

Solutions:

1. (continued)

$$\int_0^1 e^{-x^3}\,dx = \sum_0^{\infty}(-1)^k \frac{1}{(3k+1)k!}$$

Approximating this within .001 is just like approximating any other alternating series – simply make $a_{k+1} < .001$.

$$\frac{1}{(3(k+1)+1)(k+1)!} < .001 \Rightarrow (3k+4)(k+1)! > 1000$$

Experimenting with Maple, I find that $N = 4$ will do.
Therefore, accurate to within .001,

$$\int_0^1 e^{-x^3}\,dx \approx 1 - 1/4 + 1/14 - 1/60 + 1/312 \approx .80797.$$

Compare this to the approximation Maple gives: .80751.

Solutions:

2. *A Power Series for π!*

   (a) Find a power series expansion for $\frac{1}{1+x^2}$.

   Since a power series expansion for $\frac{1}{1-x}$ is

   $$\frac{1}{1-x} = P(x) = 1 + x + x^2 + x^3 + \ldots,$$

   we have

   $$\frac{1}{1+x^2} = P(-x^2) = 1 - x^2 + x^4 - x^6 + \cdots$$

   (b) Find a power series expansion for $\arctan(x)$.

   Therefore

   $$\begin{aligned}\arctan(x) &= \int \frac{1}{1+x^2}\, dx = \int 1 - x^2 + x^4 - x^6 + \cdots\, dx \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\end{aligned}$$

Solutions:

2(c) Find a power series expansion for $\frac{\pi}{4} = \arctan(1)$.

$$\begin{aligned} \arctan(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)} \\ \Rightarrow \frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)} \end{aligned}$$

2(d) Find a power series expansion for π.

$$\begin{aligned} \pi &= 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \cdot 4}{(2k+1)} \end{aligned}$$