Solutions:

1(a) $\int x \sec(x^2) \tan(x^2)\, dx$

Let $u = x^2$. Then $du = 2x\, dx$.

$$\begin{aligned}\int x \sec(x^2) \tan(x^2)\, dx &= \frac{1}{2} \int 2x \sec(x^2) \tan(x^2)\, dx \\ &= \frac{1}{2} \int \sec(u) \tan(u)\, du \\ &= \frac{1}{2} \sec(u) + C = \frac{1}{2} \sec(x^2) + C\end{aligned}$$

Solutions

1(b) $\int e^{3-2x}\,dx$

Let $u = 3 - 2x$. Then $du = -2\,dx$.

$$\begin{aligned}\int e^{3-2x}\,dx &= -\frac{1}{2}\int -2e^{3-2x}\,dx\\ &= -\frac{1}{2}\int e^{u}\,du\\ &= -\frac{1}{2}e^{u} + C = -\frac{1}{2}e^{3-2x} + C\end{aligned}$$

Solutions

1(c) $\displaystyle\int \frac{4}{x^{1/3}(1+x^{2/3})}\,dx$

Let $u = 1 + x^{2/3}$. Then $du = \frac{2}{3}x^{-1/3}\,dx = \frac{2}{3}\frac{1}{x^{1/3}}\,dx$.

$$\begin{aligned}
\int \frac{4}{x^{1/3}(1+x^{2/3})}\,dx &= 4\int \frac{1}{x^{1/3}}\frac{1}{1+x^{2/3}}\,dx \\
&= 4\cdot\frac{3}{2}\int \frac{2}{3}\frac{1}{x^{1/3}}\frac{1}{1+x^{2/3}}\,dx \\
&= 6\int \frac{1}{u}\,du \\
&= 6\ln\left|u\right| + C = 6\ln\left|1+x^{2/3}\right| + C
\end{aligned}$$

Solutions

1(d) $\displaystyle\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$

Let $u = \sqrt{x} = x^{1/2}$. Then $du = \frac{1}{2}x^{-1/2}\,dx = \frac{1}{2}\frac{1}{\sqrt{x}}\,dx$.

$$\begin{aligned}\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx &= 2\int \frac{1}{2}\frac{1}{\sqrt{x}}\sin(\sqrt{x})\,dx \\ &= 2\int \sin(u)\,du \\ &= -2\cos(u) + C = -2\cos(\sqrt{x}) + C\end{aligned}$$

Solutions

1(e) $\displaystyle\int_0^{\pi/4} \sec^2(x)e^{\tan(x)}\,dx$

Let $u = \tan(x)$. Then $du = \sec^2(x)\,dx$.
Limits of integration:

- when $x = 0$, $u = \tan(0) = 0$.
- when $x = \pi/4$, $u = \tan(\pi/4) = 1$.

$$\begin{aligned}\int_0^{\pi/4} \sec^2(x)e^{\tan(x)}\,dx &= \int_0^1 e^u\,du \\ &= e^u\Big|_0^1 \\ &= e^1 - e^0 = e - 1\end{aligned}$$

Solutions

1(f) $\displaystyle\int_{-\pi/4}^{0} \frac{\sin(x)}{\cos^2(x)}\, dx$

Let $u = \cos(x)$. Then $du = -\sin(x)\, dx$.
Limits of integration:

- when $x = -\pi/4$, $u = \cos(-\pi/4) = \frac{\sqrt{2}}{2}$.
- when $x = 0$, $u = \cos(0) = 1$.

$$\begin{aligned}
\int_{-\pi/4}^{0} \frac{\sin(x)}{\cos^2(x)}\, dx &= -1 \int_{-\pi/4}^{0} -\sin(x) \frac{1}{\cos^2(x)}\, dx \\
&= -\int_{\sqrt{2}/2}^{1} u^{-2}\, du \\
&= -(-1)u^{-1}\Big|_{\sqrt{2}/2}^{1} = 1 - \frac{2}{\sqrt{2}}
\end{aligned}$$

Solutions

1(g) $\displaystyle\int \frac{x^5}{1+x^6}\,dx$

Let $u = 1 + x^6$. Then $du = 6x^5\,dx$.

$$\begin{aligned}\int \frac{x^5}{1+x^6}\,dx &= \frac{1}{6}\int 6x^5 \frac{1}{1+x^6}\,dx \\ &= \frac{1}{6}\int \frac{1}{u}\,du \\ &= \frac{1}{6}\ln|u| + C = \frac{1}{6}\ln\left|1+x^6\right| + C\end{aligned}$$

Solutions

1(h) $\displaystyle\int \frac{x^2}{1+x^6}\,dx$

Let $u = x^3$. Then $du = 3x^2\,dx$.

$$\begin{aligned}\int \frac{x^2}{1+x^6}\,dx &= \frac{1}{3}\int 3x^2 \frac{1}{1+(x^3)^2}\,dx \\ &= \frac{1}{3}\int \frac{1}{1+u^2}\,du \\ &= \frac{1}{3}\arctan(u) + C = \frac{1}{3}\arctan(x^3) + C\end{aligned}$$

Solutions

1(i) $\int \frac{e^x}{\sqrt{1-e^x}}\,dx$

Let $u = 1 - e^x$. Then $du = -e^x\,dx$.

$$\begin{aligned}
\int \frac{e^x}{\sqrt{1-e^x}}\,dx &= -1\int -e^x \cdot \frac{1}{\sqrt{1-e^x}}\,dx \\
&= -\int \frac{1}{\sqrt{u}}\,du \\
&= -\int u^{-1/2}\,du \\
&= -2u^{1/2} + C = -2\sqrt{1-e^x} + C
\end{aligned}$$

Solutions

1(j) $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

Let $u = e^x$. Then $du = e^x\,dx$.

$$\begin{aligned}\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx &= \int e^x \cdot \frac{1}{\sqrt{1-(e^x)^2}}\,dx \\ &= \int \frac{1}{\sqrt{1-u^2}}\,du \\ &= \arcsin(u) + C = \arcsin(e^x) + C\end{aligned}$$

Solutions

1(k) $\int (x^2+4)^2 \, dx$

$$\begin{aligned} \int (x^2+4)^2 \, dx &= \int x^4 + 8x^2 + 16 \, dx \\ &= \frac{1}{5}x^5 + \frac{8}{3}x^3 + 16x + C \end{aligned}$$

Solutions

1(l) $\int x(x^2+4)^2\,dx$

Let $u = x^2+4$. Then $du = 2x\,dx$.

$$\begin{aligned}\int x(x^2+4)^2\,dx &= \frac{1}{2}\int 2x(x^2+4)^2\,dx \\ &= \frac{1}{2}\int u^2\,du \\ &= \frac{1}{2}\cdot\frac{1}{3}u^3 + C = \frac{1}{6}(x^2+4)^3 + C\end{aligned}$$

Solutions

1(m) $\int_3^4 x\sqrt{x-3}\,dx$

Let $u = x - 3$. Then $du = 1\,dx$.
Also, then $x = u + 3$.

Limits of integration:

- when $x = 3$, $u = 0$.
- when $x = 4$, $u = 1$.

$$\begin{aligned}\int_3^4 x\sqrt{x-3}\,dx &= \int_0^1 (u+3)\sqrt{u}\,du \\ &= \int_0^1 u^{3/2} + 3u^{1/2}\,du \\ &= \frac{2}{5}u^{5/2} + 2u^{3/2}\Big|_0^1 = \frac{12}{5}\end{aligned}$$

Solutions

2(a) $\int \sin(2x)\, dx$ versus $\int \sin^2(x)\, dx$

- $\int \sin(2x)\, dx$ can be done with a u-substitution.
- $\int \sin^2(x)\, dx$ can not be done with substitution (but can be done with other techniques - stay tuned!)

Let $u = 2x$. Then $du = 2\, dx$.

$$\begin{aligned} \int \sin(2x)\, dx &= \frac{1}{2}\int 2\sin(2x)\, dx \\ &= \frac{1}{2}\int \sin(u)\, du \\ &= -\frac{1}{2}\cos(u) + C = -\frac{1}{2}\cos(2x) + C \end{aligned}$$

Solutions

2(b) $\int \ln(x)\, dx$ versus $\int \frac{\ln(x)}{2x}\, dx$

- $\int \frac{\ln(x)}{2x}\, dx$ can be done with a u-substitution.
- $\int \ln(x)\, dx$ can not be done with substitution (but can be done with other techniques - stay tuned!)

Let $u = \ln(x)$. Then $du = \frac{1}{x}\, dx$.

$$\int \frac{\ln(x)}{2x}\, dx = \int \frac{1}{2} \cdot \ln(x) \frac{1}{x}$$

$$; dx$$

$$= \frac{1}{2} \int u\, du$$

$$= \frac{1}{2} \cdot \frac{1}{2} u^2 + C = \frac{1}{4}\big(\ln(x)\big)^2 + C$$

Solutions

2(c) $\displaystyle\int \frac{x^3}{1+x^8}\,dx$ versus $\displaystyle\int \frac{x^4}{1+x^8}\,dx$

- $\displaystyle\int \frac{x^3}{1+x^8}\,dx$ can be done with a u-substitution.
- $\displaystyle\int \frac{x^4}{1+x^8}\,dx$ can not be done at all.

Let $u = x^4$. Then $du = 4x^3\,dx$.

$$\begin{aligned}
\int \frac{x^3}{1+x^8}\,dx &= \frac{1}{4}\int 4 \cdot x^3 \cdot \frac{1}{1+(x^4)^2}\,dx \\
&= \frac{1}{4}\int \frac{1}{1+u^2}\,du \\
&= \frac{1}{4}\arctan(u) + C = \frac{1}{4}\arctan(x^4) + C
\end{aligned}$$

Solutions

2(d) $\int e^{-x^2}\, dx$ versus $\int xe^{-x^2}\, dx$

- $\int e^{-x^2}\, dx$ can not be done–but is important enough that we define a function to be that integral.
- $\int xe^{-x^2}\, dx$ can be done with substitution.

Let $u = -x^2$. Then $du = -2x\, dx$.

$$\begin{aligned}\int xe^{-x^2}\, dx &= -\frac{1}{2}\int -2 \cdot xe^{-x^2}\, dx \\ &= -\frac{1}{2}\int e^u\, du \\ &= -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C\end{aligned}$$

Solutions

2(e) $\int \sec(x)\, dx$ versus $\int \sec^2(x)\, dx$

- $\int \sec(x)\, dx$ can be done with a (very non-obvious) substitution.
- $\int \sec^2(x)\, dx$ doesn't require a substitution - it's just $\tan(x) + C$.

$$\begin{aligned} \int \sec(x)\, dx &= \int \sec(x) \cdot \frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)}\, dx \\ &= \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x) + \tan(x)} \end{aligned}$$

Let $u = \sec(x) + \tan(x)$. Then $du = \sec^2(x) + \sec(x)\tan(x)$.

$$\int \sec(x)\, dx = \int \frac{1}{u}\, du = \ln|u| + C = \ln|\sec(x) + \tan(x)| + C$$