Where We're Going:

- When approximating a definite integral $\mathcal{I}$, need to know how close the approximation is to the actual value of $\mathcal{I}$ – that is, need to know how big the error involved in the approximation is.
- If we can't find $\mathcal{I}$ exactly, we can't find this error exactly either.
- So we find a **bound** on how big the error is: a number that we can *guarantee* is larger than the actual size of the error. This is called an **error bound**.
- In order to have a reasonable idea of the size of the error, we'd like for our error bounds to be as close to the actual size of the error as we can find with a reasonable amount of effort.

Where We've Been:

Let f be a continuous function, and let $\mathcal{I} = \int_a^b f(x)\,dx$.

Recall: the error in using M_n to approximate $\mathcal{I} = |\mathcal{I} - M_n|$,.
Use similar notation for the error when using L_n, R_n, or T_n.

You developed two error bound formulas:

- If f is monotone on $[a, b]$ (always increasing, or always decreasing), then

$$|\mathcal{I} - L_n| \leq |R_n - L_n| \qquad \text{and} \qquad |\mathcal{I} - R_n| \leq |R_n - L_n|$$

- If f is concave up on $[a, b]$ *or* concave down on $[a, b]$, then

$$|\mathcal{I} - M_n| \leq |T_n - M_n| \qquad \text{and} \qquad |\mathcal{I} - T_n| \leq |T_n - M_n|$$

Where We've Been:

Theorem 7.1: Suppose that f'' is continuous on $[a, b]$.
Let K be any upper bound on $|f''|$ on $[a, b]$. i.e. $|f''(x)| \leq K$ for all $x \in [a, b]$.
Then

$$|\mathcal{I} - T_n| \leq \frac{K(b-a)^3}{12n^2} \qquad \text{and} \qquad |\mathcal{I} - M_n| \leq \frac{K(b-a)^3}{24n^2}$$

Note: We're using the word *bound* twice.

- $\frac{K(b-a)^3}{12n^2}$ and $\frac{K(b-a)^3}{24n^2}$ are error bounds (an upper bound on the size of the error)
- K is an upper bound on the size of $|f''(x)|$ on the interval $[a, b]$.

Example:

Let $\mathcal{I} = \int_0^{10} e^{\cos(x)}\, dx$**. How close does** M_{50} **approximate** I**?**

- Use Maple to find M_{50}.
- How good an approximation is it?
- The *actual error* is $|\mathcal{I} - M_{50}|$.
- If Maple won't calculate, can't know the actual error exactly.
- Plug $a = 0$, $b = 10$, $n = 50$ into $|\mathcal{I} - M_{50}| \leq \dfrac{K(b-a)^3}{24n^2}$; simplify.
- To find K: Use Maple to look at the graph of $|f''(x)|$ over the interval, and pick a number close to but bigger than the the absolute value of the second derivative ever is on $[0, 10]$. (See next slide). This is your choice for K.
- Substitute into error bound formula to find an upper bound on the error, that you hope is reasonably close to the actual error.

Using Maple to find a good choice for K:

- Type in the function (in our example, `exp(cos(x))`)
- Control-click on output (the function).
  `Differentiate - with respect to x`
- Control-click on new output (the derivative).
  `Differentiate - with respect to x`
- Click on absolute-value bars in Expression Palette. Replace the a with the 2nd derivative by clicking on Command (squiggle) L, and entering the line number of the second derivative in the pop-up menu.
- Control-click on the output (absolute value of the 2nd deriv).
  `Plots-Plot Builder`, enter $[0, 10]$
- Observe the graph. Pick a number that is higher than but close to $|f''(x)|$ on all of $[a, b]$

  If you want to check whether a potential choice for K is in fact larger than $|f''(x)|$
  - Example: To check $|f''(x)| \leq 2.75$ over $[a, b]$, type `2.75`. Highlight, drag onto graph of $|f''(x)|$. If the horizontal line is always above the graph of the second derivative, can use that number as K.

In Class Work

Let $\mathcal{I} = \int_0^4 \arctan\left(x^2 \sin(x)\right)\, dx$

1. Use Maple to calculate T_{100}.
   `Tools-Tutors-Calculus-Single Variable - Riemann Sums`

2. How close is T_{100} to the actual value of $\mathcal{I}$?

3. Determine how many subintervals n you need to use in order for T_n to approximate $\mathcal{I}$ within 0.0001. Find that T_n using Maple.

Solutions

Let $\mathcal{I} = \int_0^4 \arctan\left(x^2 \sin(x)\right)\, dx$

1. Use Maple to calculate T_{100}.

$$T_{100} \approx 1.578246303.$$

2. How close is T_{100} to the actual value of $\mathcal{I}$?

$$|\mathcal{I} - T_n| \leq \frac{K(b-a)^3}{12n^2} = \frac{K(4-0)^3}{(12)(100)^2} = \frac{K}{1875} \approx 0.0005333K$$

Looking at the graph of $|f''(x)|$ over $[0,4]$, $K = 65$ will work. Thus

$$|\mathcal{I} - T_n| \leq \frac{65}{1875} \approx 0.0347.$$

T_{100} is within 0.0347 of the actual value of $\mathcal{I}$.
That is,

$$T_{100} - 0.0347 \leq \mathcal{I} \leq T_{100} + 0.0347$$

Solutions

Let $\mathcal{I} = \int_0^4 \arctan\left(x^2 \sin(x)\right)\, dx$

3. Determine how many subintervals n you need to use in order for T_n to approximate $\mathcal{I}$ within 0.0001. Find T_n using Maple.

Still know: $a = 0$, $b = 4$, $K = 65$. But this time, we don't know n.

$$|\mathcal{I} - T_n| \leq \frac{K(b-a)^3}{12n^2} = \frac{65(4-0)^3}{12n^2} \overset{\text{simplify}}{=} \frac{1040}{3n^2} \approx \frac{346.67}{n^2}$$

Want: $|\mathcal{I} - T_n| \leq 0.0001$ **Know:** $|\mathcal{I} - T_n| \leq \dfrac{1040}{3n^2}$

If find n so that $\dfrac{1040}{3n^2} \leq 0.0001$, then will have

$$|\mathcal{I} - T_n| \leq \frac{1040}{3n^2} \leq 0.0001 \Rightarrow |\mathcal{I} - T_n| \leq 0.0001$$

Solutions (continued)

Let $\mathcal{I} = \int_0^4 \arctan\left(x^2 \sin(x)\right)\, dx$

3. Determine how many subintervals n you need to use in order for T_n to approximate $\mathcal{I}$ within 0.0001. Find T_n using Maple.

   Need to find n so that : $\dfrac{1040}{3n^2} \leq 0.0001$

$$\begin{aligned}
\frac{1040}{3n^2} &\leq \frac{1}{10000} \\
\frac{10400000}{3} &\leq n^2 \\
\sqrt{\frac{10400000}{3}} &\leq n \\
1861.8 &\leq n
\end{aligned}$$

Using $n = 1862$, $T_{1862} \approx 1.57826$