Recall: Facts emblazoned on your soul

- $\int_0^a \frac{1}{x^p}\,dx$ converges if $p < 1$ (if $a = 1$, to $\frac{1}{1-p}$), diverges for $p \geq 1$.
- $\int_a^\infty \frac{1}{x^p}\,dx$ diverges for $p \leq 1$, converges if $p > 1$ (if $a = 1$, to $\frac{1}{p-1}$).

Note: I said it converges to $\frac{1}{1-p}$ before break - denom is reversed.

- If $\lim_{x\to\infty} f(x) \neq 0$, $\int_a^\infty f(x)\,dx$ **must** diverge.
- If $\lim_{x\to\infty} f(x) = 0$, $\int_a^\infty f(x)\,dx$ *may* converge (like $\int_a^\infty \frac{1}{x^p}\,dx$ with $p > 1$), or it *may* diverge (like $\int_a^\infty \frac{1}{x^p}\,dx$ with $p \leq 1$).

In other words – $\lim_{x\to\infty} f(x) = 0$ tells you **nothing** about the convergence or divergence of $\int_a^\infty f(x)\,dx$.

Also Recall: The Comparison Test

Suppose $0 \leq f(x) \leq g(x)$

- If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ must also converge,
- but if $\int_a^\infty f(x)\,dx$ converges, that tells you **nothing** about whether or not $\int_a^\infty g(x)\,dx$ converges.

- Similarly, if $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ must also diverge,
- but if $\int_a^\infty g(x)\,dx$ diverges, that tells you **nothing** about whether or not $\int_a^\infty f(x)\,dx$ diverges.

Examples:

What would be your *first choice* of comparison integrals? (Note: not all will actually be helpful)

1. $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}-1}$
2. $\displaystyle\int_1^\infty \frac{1}{x^2+x}$
3. $\displaystyle\int_0^\infty \frac{1}{e^x+2}$
4. $\displaystyle\int_1^\infty \frac{1}{x+\cos(x)}$

Examples:

What would be your *first choice* of comparison integrals? (Note: not all will actually be helpful)

1. $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}-1}$ Try comparing to $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}}\,dx$
2. $\displaystyle\int_1^\infty \frac{1}{x^2+x}$
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2. $\displaystyle\int_1^\infty \frac{1}{x^2+x}$ Try comparing to either $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ or to $\displaystyle\int_1^\infty \frac{1}{x}\,dx$
3. $\displaystyle\int_0^\infty \frac{1}{e^x+2}$
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What would be your *first choice* of comparison integrals? (Note: not all will actually be helpful)

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4. $\displaystyle\int_1^\infty \frac{1}{x+\cos(x)}$ Try comparing to $\displaystyle\int_1^\infty \frac{1}{x}\,dx$

In Class Work

Use the Comparison Test to determine whether each of the following improper integrals converges or diverges. In each case be sure to discuss with your group whether your comparison integral is in fact helpful and why.

1. $\displaystyle\int_2^\infty \frac{1}{x^3+2}\,dx$
2. $\displaystyle\int_5^\infty \frac{1}{\sqrt{x}-2}\,dx$
3. $\displaystyle\int_2^\infty \frac{2}{\sqrt{x}+x^2}\,dx$
4. $\displaystyle\int_0^\infty \frac{2}{\sqrt{x}+x^2}\,dx$
5. $\displaystyle\int_5^\infty \frac{5x-4}{x^7+8x}\,dx$
6. $\displaystyle\int_7^\infty \frac{2}{3x^4-6}\,dx$

Using Polynomials to Approximate Other Functions

By insisting that the first six derivatives match at $x = 0$, we can create a 6th degree polynomial $P_6(x)$ that approximates $\cos(x)$ fairly well.
That is, by forcing $P_6(x)$ to satisfy

$$\begin{aligned} P_6(0) &= \cos(0) & P_6^{(4)}(0) &= \cos(0) \\ P_6'(0) &= -\sin(0) & P_6^{(5)}(0) &= -\sin(0) \\ P_6''(0) &= -\cos(0) & P_6^{(6)}(0) &= -\cos(0) \\ P_6'''(0) &= \sin(0) \end{aligned}$$

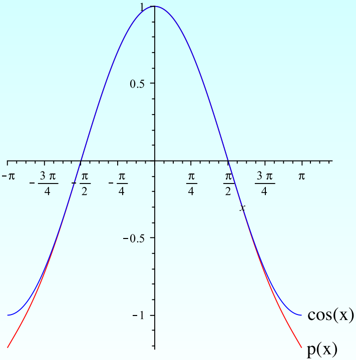
We can find $P_6(x)$ so that

$$P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \text{ approximates } \cos(x) \text{ near } 0.$$

Using Polynomials to Approximate Other Functions

$$P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \text{ approximates } \cos(x) \text{ near } 0.$$

(For those who've seen them before, this is a Taylor (or MacLaurin) polynomial.)



Using Polynomials to Approximate Other Functions

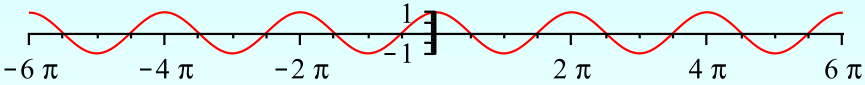
We can use these *Taylor Polynomial* approximations to approximate

- The value of a more complicated function at a particular point
- The derivative of a more complicated function at a point
- The integral of a more complicated function at a point

Using Polynomials to Approximate Other Functions

The more derivatives *at* $x = 0$ we match (the higher n is), the better $P_n(x)$ does at approximating $\cos(x)$ *away* from $x = 0$.

The following is a graph of $P_{50}(x)$:



Solutions:

Determine whether each of the following improper integrals converges or diverges.

1. $\displaystyle\int_2^\infty \frac{1}{x^3+2}\,dx$ First choice: compare to $\displaystyle\int_2^\infty \frac{1}{x^3}\,dx$. Useful?

$$\begin{aligned} x^3+2 &> x^3 \\ 0 < \frac{1}{x^3+2} &< \frac{1}{x^3} \\ 0 \leq \int_2^\infty \frac{1}{x^3+2}\,dx &\leq \int_2^\infty \frac{1}{x^3}\,dx \end{aligned}$$

$\displaystyle\int_2^\infty \frac{1}{x^3}\,dx = \int_a^\infty \frac{1}{x^p}\,dx$ with $p > 1$, so converges (to $\frac{1}{2}$).

Thus $\displaystyle 0 \leq \int_2^\infty \frac{1}{x^3+2}\,dx \leq \frac{1}{2}$, so it too must converge.

Solutions:

2. $\displaystyle\int_5^\infty \frac{1}{\sqrt{x}-2}\,dx$ First choice: compare to $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}}\,dx$. Useful?

$$\begin{aligned} 0 < \sqrt{x}-2 &< \sqrt{x} \text{ on } [5,\infty) \\ \frac{1}{\sqrt{x}-2} &> \frac{1}{\sqrt{x}} > 0 \text{ on } [5,\infty) \\ \int_5^\infty \frac{1}{\sqrt{x}-2}\,dx &\geq \int_5^\infty \frac{1}{\sqrt{x}}\,dx \geq 0 \end{aligned}$$

$\displaystyle\int_5^\infty \frac{1}{\sqrt{x}}\,dx = \int_5^\infty \frac{1}{x^p}\,dx$ with $p \leq 1$, so diverges to infinity.

Since $\displaystyle\int_5^\infty \frac{1}{\sqrt{x}}\,dx$ diverges (to infinity), and $\displaystyle\int_5^\infty \frac{1}{\sqrt{x}-2}\,dx$ is at least as large, it too must diverge.

Solutions:

3. $\displaystyle\int_2^\infty \frac{2}{\sqrt{x}+x^2}\,dx$ Compare to $\displaystyle\int_2^\infty \frac{2}{\sqrt{x}}\,dx$? $\displaystyle\int_2^\infty \frac{2}{x^2}\,dx$? Which?

$$\begin{aligned} \sqrt{x}+x^2 &> \sqrt{x} \text{ and } x^2 \geq 0 \\ 0 \leq \frac{2}{\sqrt{x}+x^2} &< \frac{2}{\sqrt{x}} \text{ and } \frac{2}{x^2} \\ 0 \leq \int_2^\infty \frac{2}{\sqrt{x}+x^2}\,dx &\leq \int_2^\infty \frac{2}{\sqrt{x}}\,dx \text{ and } \int_2^\infty \frac{2}{x^2}\,dx \end{aligned}$$

$\displaystyle\int_2^\infty \frac{2}{\sqrt{x}}\,dx$ diverges to infinity, since $p = \dfrac{1}{2} < 1$.

Knowing that $\displaystyle\int_2^\infty \frac{2}{\sqrt{x}+x^2}\,dx \leq \infty$ is **not** helpful!

$\displaystyle\int_2^\infty \frac{2}{x^2}\,dx$ converges, since $p = 2 > 1$.

Thus $0 \leq \displaystyle\int_2^\infty \frac{2}{\sqrt{x}+x^2}\,dx \leq$ a finite number, so it converges.

Solutions:

4. $\displaystyle\int_0^\infty \frac{2}{\sqrt{x}+x^2}\,dx$ Compare to $\displaystyle\int_0^\infty \frac{2}{\sqrt{x}}\,dx$? $\displaystyle\int_0^\infty \frac{2}{x^2}\,dx$?

Improper at both ends–split into 2 integrals.
Can split at any positive x-value - I choose $x = 2$.

$$\int_0^\infty \frac{2}{\sqrt{x}+x^2}\,dx = \int_0^2 \frac{2}{\sqrt{x}+x^2}\,dx + \int_2^\infty \frac{2}{\sqrt{x}+x^2}\,dx$$

Just found $\int_2^\infty \frac{2}{\sqrt{x}+x^2}\,dx$ converges, comparing to $\int_2^\infty \dfrac{2}{x^2}\,dx$.
How about $\int_0^2 \frac{2}{\sqrt{x}+x^2}\,dx$? Using the same inequalities as before,

$$0 \leq \int_0^2 \frac{2}{\sqrt{x}+x^2}\,dx \quad \leq \quad \int_0^2 \frac{2}{\sqrt{x}}\,dx \text{ and } \int_0^2 \frac{2}{x^2}\,dx$$

Comparison integral on the far right diverges – useless comparison.
Comparison integral on the near right converges – useful comparison.
Putting it all together, we're adding up two finite pieces, and so the whole thing also converges.

Solutions:

5. $\displaystyle\int_5^\infty \frac{5x-4}{x^7+8x}\,dx$ First choice: maybe compare to $\displaystyle\int_5^\infty \frac{5x}{x^7}\,dx$?

$$5x-4 \le 5x$$

$$x^7+8x \ge x^7 \quad\Rightarrow\quad \frac{1}{x^7+8x} \le \frac{1}{x^7}$$

$$\therefore\ \frac{5x-4}{x^7+8x} \le \frac{5x}{x^7}$$

$$\Rightarrow \int_5^\infty \frac{5x-4}{x^7+8x}\,dx \le \int_5^\infty \frac{5x}{x^7}\,dx = 5\int_5^\infty \frac{1}{x^6}\,dx$$

$5\displaystyle\int_5^\infty \frac{1}{x^6}\,dx$ converges, since $p=6>1$

Thus $0 \le \displaystyle\int_5^\infty \frac{5x-4}{x^7+8x}\,dx \le$ a finite number, so it converges.

Solutions:

6. $\displaystyle\int_7^\infty \frac{2}{3x^4-6}\,dx$ First choice: Compare to $\displaystyle\frac{2}{3}\int_7^\infty \frac{1}{x^4}\,dx$?

$$\begin{aligned} 3x^4-6 &\le 3x^4 \\ \Rightarrow \frac{2}{3x^4-6} &\ge \frac{2}{3x^4} \\ \Rightarrow \int_7^\infty \frac{2}{3x^4-6}\,dx &\ge \frac{2}{3}\int_7^\infty \frac{1}{x^4}\,dx \end{aligned}$$

$\frac{2}{3}\displaystyle\int_7^\infty \frac{1}{x^4}\,dx$ converges, since $p=4>1$.

Knowing that $\displaystyle\int_7^\infty \frac{2}{3x^4-6}\,dx \ge$ a finite number is **not** helpful!

We must find something else to compare to.

Solutions:

6. $\displaystyle\int_7^\infty \frac{2}{3x^4-6}\,dx$ First choice, $\displaystyle\frac{2}{3}\int_7^\infty \frac{1}{x^4}\,dx$, didn't work

We must find something else to compare to.

Dropping the constant half of the sum in the denominator won't work.

Next option: Try comparing that constant to a multiple of the other term in the denominator.

$$\begin{aligned}
6 \le x^4 \text{ for all } x \ge 7 \quad &\Rightarrow \quad -6 \ge -x^4 \text{ for all } x \ge 7\\
\Rightarrow 3x^4 - 6 \quad &\ge \quad 3x^4 - x^4 = 2x^4 \text{ for all } x \ge 7\\
\Rightarrow \frac{2}{3x^4-6} \quad &\le \quad \frac{2}{2x^4} = \frac{1}{x^4} \text{ for all } x \ge 7\\
\Rightarrow \int_7^\infty \frac{2}{3x^4-6}\,dx \quad &\le \quad \int_7^\infty \frac{1}{x^4}\,dx
\end{aligned}$$

$\displaystyle\int_7^\infty \frac{1}{x^4}\,dx$ converges, since $p = 4 > 1$.

$\displaystyle 0 \le \int_7^\infty \frac{2}{3x^4-6}\,dx \le$ a finite #, so $\displaystyle\int_7^\infty \frac{2}{3x^4-6}\,dx$ must converge