Recall:

Let $\{a_k\}_{k=0}^{\infty} = \{a_0, a_1, a_2, a_3, \dots\}$ be any sequence.

- Associated **sequence of partial sums**:

$$\{S_n\}_{n=0}^{\infty} = \left\{\sum_{k=0}^{n} a_k\right\}_{n=0}^{\infty} = \{a_0, a_0 + a_1, a_0 + a_1 + a_2, \dots, \}$$

- We define $\lim\limits_{n\to\infty} S_n$ to be the **series** associated with the **sequence of terms** $\{a_k\}$. We denote the series $\sum\limits_{k=0}^{\infty} a_k$.

- The series converges $\iff \lim\limits_{n\to\infty} S_n$ exists $\iff \lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} a_k$ exists $\Leftrightarrow$ $\lim\limits_{n\to\infty} (a_0 + a_1 + a_2 + \dots + a_n)$ exists.

Example:

For the series $\displaystyle\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k$:

- the sequence of terms $\left\{a_k\right\}$ is

$$\left\{\left(\frac{1}{4}\right)^k\right\}_{k=0}^{\infty} = \{1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}, \ldots\}$$

- The associated sequence of partial sums $\left\{S_n\right\}$ is

$$\{1, 1+\frac{1}{4}, 1+\frac{1}{4}+\frac{1}{16}, 1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}, 1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}, \ldots\}$$

$$= \{1, \frac{5}{4}, \frac{21}{16}, \frac{85}{64}, \frac{341}{256}, \ldots\}$$

- This series **converges** if $\displaystyle\lim_{n\to\infty} S_n$ exists and is finite; **diverges** otherwise.

Example

Since $\displaystyle\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k$ is a geometric series, with $r = \dfrac{1}{4}$, and since $|r| < 1$, this series converges.

Recall: A **geometric series** is a series of the form
$1 + r + r^2 + r^3 + \cdots = \displaystyle\sum_{k=0}^{\infty} r^k$

In fact, our series converges to $\dfrac{1}{1-\frac{1}{4}} = \dfrac{4}{3}$.

Goals:

1. Determine whether a series $\sum a_k$ converges or diverges.
2. If it converges, find the limit (that is, the value of the series) exactly, if possible.
3. If it converges but we can't find the limit exactly, be able to approximate it.

Methods we have so far:

Given a series $\displaystyle\sum_{k=M}^{\infty} a_k$,

- **Is it a geometric series?** If so, we can determine whether or not it converges, and if so, exactly what it converges to.
- **nth Term Test:** If $\displaystyle\lim_{k\to\infty} a_k \neq 0$, the series diverges. If $\displaystyle\lim_{k\to\infty} a_k = 0$, inconclusive.
- p **-Test:** Series of the form $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k^p}$ where $p > 1$ converge; if $p \leq 1$ the series diverges.

Solutions:

1(a) $\displaystyle\sum_{k=0}^{\infty} \frac{4}{3^k}$

(i) Find a_2 and a_3.

$$a_2 = \frac{4}{3^2} = \frac{4}{9} \qquad a_3 = \frac{4}{3^3} = \frac{4}{27}.$$

(ii) Find S_2 and S_3.

$$\begin{aligned} S_2 &= \sum_{k=0}^{2} \frac{4}{3^k} = 4 + \frac{4}{3} + \frac{4}{9} = \frac{52}{9} \approx 5.78 \\ S_3 &= \sum_{k=0}^{3} \frac{4}{3^k} = 4 + \frac{4}{3} + \frac{4}{9} + \frac{4}{27} = \frac{52}{9} + \frac{4}{27} = \frac{160}{27} \approx 5.93 \end{aligned}$$

(iii) Geometric series with ratio $r = \dfrac{1}{3}$. $|r| < 1 \Longrightarrow$ series converges, to

$$S = \sum_{k=0}^{\infty} \frac{4}{3^k} = 4\sum_{k=0}^{\infty} \frac{1}{3^k} = 4 \cdot \frac{1}{1 - 1/3} = 4 \cdot \frac{3}{2} = 6.$$

Solutions:

1(b) $\displaystyle\sum_{k=0}^{\infty} \frac{3^k}{(-4)^k}$

(i) Find a_2 and a_3.

$$a_2 = \frac{3^2}{(-4)^2} = \frac{9}{16} \qquad a_3 = \frac{3^3}{(-4)^3} = -\frac{27}{64}.$$

(ii) Find S_2 and S_3.

$$S_2 = \sum_{k=0}^{2} \frac{3^k}{(-4)^k} = 1 - \frac{3}{4} + \frac{9}{16} = \frac{13}{16} \approx 0.813$$

$$S_3 = \sum_{k=0}^{3} \frac{3^k}{(-4)^k} = 1 - \frac{3}{4} + \frac{9}{16} - \frac{27}{64} = \frac{13}{16} - \frac{27}{64} = \frac{25}{64} \approx 0.391$$

(iii) $\displaystyle\sum_{k=0}^{\infty} \frac{3^k}{(-4)^k} = \sum_{k=0}^{\infty} \left(-\frac{3}{4}\right)^k$. Geometric series with ratio $r = -\frac{3}{4}$.

$|r| < 1 \implies$ series converges to: $S = \dfrac{1}{1-(-3/4)} = \dfrac{1}{7/4} = 4/7$

Solutions:

1(c) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{5^k}$

(i) Find a_2 and a_3:

$$a_2 = \frac{1}{5^2} = \frac{1}{25} \qquad a_3 = \frac{1}{5^3} = \frac{1}{125}$$

(ii) Find S_2 and S_3.

$$S_2 = \sum_{k=2}^{2} \frac{1}{5^2} = \frac{1}{25} = 0.04$$

$$S_3 = \sum_{k=2}^{3} \frac{1}{5^k} = \frac{1}{25} + \frac{1}{125} = \frac{6}{125} = \frac{25}{64} = 0.48$$

(iii) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{5^k}$

- Geometric series; $r = \dfrac{1}{5}$. Since $|r| < 1$, series converges.
- **BUT** the series doesn't converge to $\dfrac{1}{1-r}$, since not starting at $k = 0$.

Solutions:

1(c) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{5^k}$ (Continued)

(iii) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{5^k}$ convergent geometric series, but starts at $k = 2$.

- To see what it converges to, write out the first several terms

$$
\begin{aligned}
S &= \sum_{k=2}^{\infty} \frac{1}{5^k} = \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \dots \\
&= \frac{1}{5^2}\left(1 + \frac{1}{5} + \frac{1}{5^2} + \dots\right) \\
&= \frac{1}{5^2} \sum_{k=0}^{\infty} \frac{1}{5}^k \\
&= \frac{1}{5^4}\left(\frac{1}{1 - 1/5}\right) \\
&= \frac{1}{5^2} \cdot \frac{5}{4} = \frac{1}{20}
\end{aligned}
$$

Solutions:

1(d) $\displaystyle\sum_{k=1}^{\infty} \frac{2k^2-3}{5k^2+6k}$

(i) $a_2 = \dfrac{8-3}{20+12} = \dfrac{5}{32}$ $\qquad a_3 = \dfrac{18-3}{45+18} = \dfrac{15}{63}$

(ii)
$$S_2 = \sum_{k=1}^{2} \frac{2k^2-3}{5k^2+6k} = \frac{-1}{11} + \frac{5}{32} \approx 0.065341$$
$$S_3 = \sum_{k=1}^{3} \frac{2k^2-3}{5k^2+6k} = \frac{-1}{11} + \frac{5}{32} + \frac{15}{63} \approx 0.303436$$

(iii) Not a geometric series. Try the *n*th term test.

$$\lim_{k\to\infty} \frac{2k^2-3}{5k^2+6k} \left(\frac{\infty}{\infty}\right) \stackrel{\text{l'Hôp}}{=} \lim_{k\to\infty} \frac{4k}{10k+6} \left(\frac{\infty}{\infty}\right) \stackrel{\text{l'Hôp}}{=} \lim_{k\to\infty} \frac{4}{10} \neq 0$$

Sequence of terms a_k converges to $\frac{2}{5}$.

Series $\sum a_k$ diverges by *n*th term test

Solutions:

1(e) $\displaystyle\sum_{k=98}^{\infty} \frac{3^k + \sin(k)}{\cos(k) + 5}$

- Not a geometric series. Try the nth term test.
  - **Note:** *Always* try the nth term test. Won't tell you that a series converges, but it can sometimes tell you a series diverges.

$$\lim_{k\to\infty} 3^k + \sin(k) = \infty \qquad \lim_{k\to\infty} \cos(k) + 5 \text{ doesn't exist.}$$

l'Hôpital's rule does not apply. How can we figure out this limit?

The numerator is increasing without bound. The denominator is always between 4 and 6. This limit will be infinity.

The *sequence of terms* diverges, and so obviously the the sequence of terms doesn't approach 0.

When the *sequence of terms* doesn't converge to 0, the nth term test guarantees that the *series* $\sum_{k=98}^{\infty} a_k$ diverges.

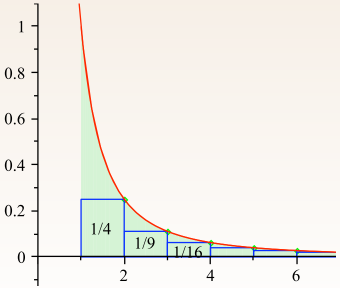
Solutions:

2. Determine whether the following series converge or diverge, by drawing a picture that compares each series to an improper integral

(a) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2} = \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$

Notice that the nth term test is inconclusive

Informally represent our sum as a right sum with $\Delta x = 1$.



Our series (the right sum) is less than the integral. Since the area under the curve is finite, this sum must also be finite!

Notice:

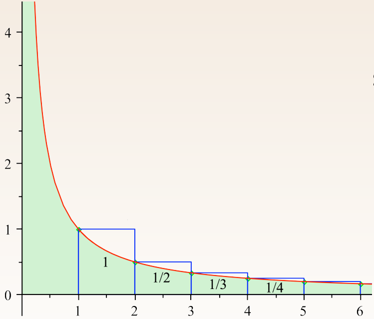
- The sequence of terms a_k converges to 0
- The series $\sum a_k$ converges, but not to 0.

Solutions:

2(b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$

The *n*th term test is inconclusive

Represent our sum as a left sum with $\Delta x = 1$



Our series (the left sum) is greater than the integral. We know this integral is infinite, so the sum is infinite as well.

Notice:

- The sequence of terms a_k converges to 0
- The series $\sum a_k$ diverges.
- This series is called the harmonic series!