Example:

Find $\int \sqrt{9+(4x+5)^2}\, dx$.

You don't know how to do this integral.
But it doesn't look very hard, so you think maybe you're just missing something.
You turn to Maple, and it produces:

$$\frac{x}{2}\sqrt{9+16x^2}+\frac{9}{8}\text{arcsinh}\left(\frac{4x}{3}\right).$$

Since you don't know anything about the arcsinh (inverse hyperbolic sine) function, you know you really didn't know how to do that integral.

You could just proceed using this result, but it's totally meaningless to you.

Example:

Find $\int \sqrt{9+(4x+5)^2}\,dx$

An alternative: Check whether integral tables will help.
Some headings:

Forms Involving $\mathbf{a+bu}$

$\vdots$

Forms Involving $\mathbf{(a+bu)^2}$

$\vdots$

Forms Involving $\sqrt{\mathbf{a+bu}}$

$\vdots$

Forms Involving $\sqrt{\mathbf{a^2+u^2}}, \mathbf{a>0}$

$\vdots$

Forms Involving $\sqrt{\mathbf{a^2-u^2}}, \mathbf{a>0}$

$\vdots$

Forms Involving $\sqrt{\mathbf{u^2-a^2}}, \mathbf{a>0}$

$\vdots$

Forms Involving $\sqrt{\mathbf{2au-u^2}}$

$\vdots$

Forms Involving $\sin(\mathbf{u})$ or $\cos(\mathbf{u})$

$\vdots$

Example:

Find $\int \sqrt{9 + (4x+5)^2}\, dx$

Forms Involving $\sqrt{\mathbf{a^2 + u^2}}, \mathbf{a > 0}$

21. $\int \sqrt{a^2 + u^2}\, du = \frac{1}{2} u \sqrt{a^2 + u^2} + \frac{1}{2} a^2 \ln \left| u + \sqrt{a^2 + u^2} \right| + c$

22. $\int u^2 \sqrt{a^2 + u^2}\, du = \frac{1}{8} u (a^2 + 2u^2) \sqrt{a^2 + u^2} - \frac{1}{8} a^4 \ln \left| u + \sqrt{a^2 + u^2} \right| + c$

23. $\int \frac{\sqrt{a^2 + u^2}}{u}\, du = \sqrt{a^2 + u^2} - a \ln \left| \frac{a + \sqrt{a^2 + u^2}}{u} \right| + c$

24. $\int \frac{\sqrt{a^2 + u^2}}{u^2}\, du = \ln \left| u + \sqrt{a^2 + u^2} \right| - \frac{\sqrt{a^2 + u^2}}{u} + c$

25. $\int \frac{1}{\sqrt{a^2 + u^2}}\, du = \ln \left| u + \sqrt{a^2 + u^2} \right| + c$

26. $\int \frac{u^2}{\sqrt{a^2 + u^2}}\, du = \frac{1}{2} u \sqrt{a^2 + u^2} - \frac{1}{2} a^2 \ln \left| u + \sqrt{a^2 + u^2} \right| + c$

Example:

Find $\int \sqrt{9 + (4x+5)^2}\, dx$

Forms Involving $\sqrt{\mathbf{a^2 + b^2}}, \mathbf{a > 0}$

21. $\int \sqrt{a^2 + u^2}\, du = \frac{1}{2} u \sqrt{a^2 + u^2} + \frac{1}{2} a^2 \ln \left| u + \sqrt{a^2 + u^2} \right| + c$

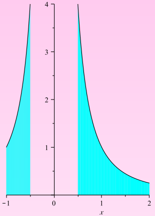
Let $a = 3$ and $u = 4x + 5 \Rightarrow du = 4\, dx$

$$\int \sqrt{9 + (4x+5)^2}\, dx = \frac{1}{4} \int \sqrt{a^2 + u^2}\, du.$$

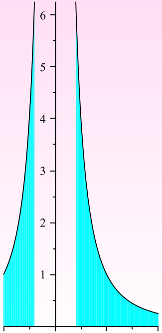
Thus using the formula from the integral table,

$$\begin{aligned} \int \sqrt{9 + (4x+5)^2}\, dx &= \frac{1}{4} \left[\frac{4x+5}{2} \sqrt{9 + (4x+5)^2} \right. \\ &\qquad \left. + \frac{9}{2} \ln \left| 4x + 5 + \sqrt{9 + (4x+5)^2} \right| \right] + c. \end{aligned}$$

Can we make sense of $\int_{-1}^{2} \frac{1}{x^2}\, dx$?

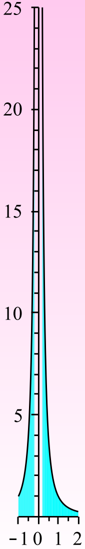


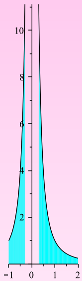
$$\int_{-1}^{-.5} \frac{1}{x^2}\, dx + \int_{.5}^{2} \frac{1}{x^2}\, dx = 2.5$$



$$\int_{-1}^{-.4} \frac{1}{x^2}\, dx + \int_{.4}^{2} \frac{1}{x^2}\, dx = 3.5$$

Making sense of $\int_{-1}^{2} \frac{1}{x^2}\,dx$





$$\int_{-1}^{-.3} \frac{1}{x^2}\,dx + \int_{.3}^{2} \frac{1}{x^2}\,dx = 5.1666666$$

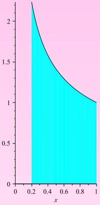
$$\int_{-1}^{-.2} \frac{1}{x^2}\,dx + \int_{.2}^{2} \frac{1}{x^2}\,dx = 8.5$$

Making sense of $\int_{-1}^{2} \frac{1}{x^2}\, dx$

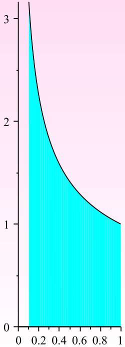
What happens as we move in closer to $x = 0$?

R	$\int_{-1}^{-R} \frac{1}{x^2}\, dx + \int_{R}^{2} \frac{1}{x^2}\, dx$
0.5	Signed Area = 2.5
0.4	Signed Area = 3.5
0.3	Signed Area = 5.1666666
0.2	Signed Area = 8.5
0.1	Signed Area = 18.5
0.01	Signed Area = 198.5
0.001	Signed Area = 1998.5
0.0001	Signed Area = 19998.5
0.00001	Signed Area = 199998.5
0.000001	Signed Area = 1999998.5
0.0000001	Signed Area = 19999998.5

Making sense of $\int_0^1 \frac{1}{\sqrt{x}}\,dx$:



$$\int_{.2}^1 \frac{1}{\sqrt{x}}\,dx = 1.105572809$$



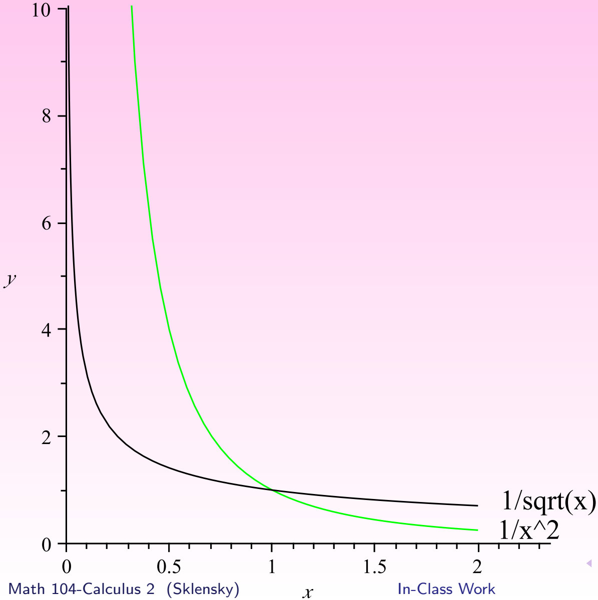
$$\int_{.1}^1 \frac{1}{\sqrt{x}}\,dx = 1.367544468$$

Making sense of $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$:

Now what happens as we move in closer to $x = 0$?

R	$\displaystyle\int_R^1 \frac{1}{\sqrt{x}}\,dx$
0.2	Signed Area $= 1.105572809$
0.1	Signed Area $= 1.367544468$
0.01	Signed Area $= 1.8$
0.001	Signed Area $= 1.936754447$
0.0001	Signed Area $= 1.98$
0.00001	Signed Area $= 1.993675445$
0.000001	Signed Area $= 1.998$

Comparing $\frac{1}{x^2}$ and $\frac{1}{\sqrt{x}}$ near $x = 0$



Definition:

If f is continuous on the interval $[a, b)$, but $|f(x)| \to \infty$ as $x \to b^-$, we define the **improper integral** of f on $[a, b]$ by

$$\int_a^b f(x)\, dx = \lim_{R \to b^-} \int_a^R f(x)\, dx.$$

Similarly, if f is continuous on the interval $(a, b]$, but $|f(x)| \to \infty$ as $x \to a^+$, we define the improper integral

$$\int_a^b f(x)\, dx = \lim_{R \to a^+} \int_R^b f(x)\, dx.$$

In either case, if the limit exists and equals some finite value L, we say that the improper integral **converges** to L. If the limit does not exist (whether because it equals $\pm\infty$ or because for some other reason), we say that the improper integral **diverges**.

In Class Work

Determine whether the following improper integrals converge or diverge. For those that converge, what value do they converge to?

1. $\displaystyle\int_0^1 \frac{1}{x^{1/3}}\,dx$
2. $\displaystyle\int_0^{\pi/2} \frac{\sin(x)}{\sqrt{\cos(x)}}\,dx$
3. $\displaystyle\int_0^1 \frac{1}{x}\,dx$
4. $\displaystyle\int_0^1 \frac{1}{x^3}\,dx$

Solutions:

Determine whether the following improper integrals converge or diverge. For those that converge, what value do they converge to?

1. $\int_0^1 \frac{1}{x^{1/3}}\,dx$

$$\begin{aligned}
\int_0^1 \frac{1}{x^{1/3}}\,dx &= \lim_{R\to 0^+} \int_R^1 x^{-1/3}\,dx \\
&= \lim_{R\to 0^+} \frac{3}{2}x^{2/3}\Big|_R^1 \\
&= \frac{3}{2}\lim_{R\to 0^+} 1 - R^{2/3} \\
&= \frac{3}{2}
\end{aligned}$$

Thus the improper integral converges, to $\frac{3}{2}$.

Solutions:

2. $\displaystyle\int_0^{\pi/2} \frac{\sin(x)}{\sqrt{\cos(x)}}\, dx$

   Let $u = \cos(x)$, so $du = -\sin(x)\, dx$.
   $x = 0 \Rightarrow u = 1$, $x = \pi/2 \Rightarrow u = 0$.

$$
\begin{aligned}
\int_0^{\pi/2} \frac{\sin(x)}{\sqrt{\cos(x)}}\, dx &= -\int_1^0 \frac{1}{\sqrt{u}}\, du \\
&= \lim_{R\to 0^+} \int_R^1 u^{-1/2}\, du \\
&= \lim_{R\to 0^+} 2\sqrt{u}\Big|_R^1 \\
&= \lim_{R\to 0^+} 2 - 2\sqrt{R} \\
&= 2
\end{aligned}
$$

Thus this improper integral converges, to 2.

Solutions:

3. $\displaystyle\int_0^1 \frac{1}{x}\,dx$

$$\begin{aligned}\int_0^1 \frac{1}{x}\,dx &= \lim_{R\to 0^+} \int_R^1 \frac{1}{x}\,dx \\ &= \lim_{R\to 0^+} \ln(x)\Big|_R^1 \\ &= \lim_{R\to 0^+} 0 - \ln(R) \\ &= -(-\infty) = \infty\end{aligned}$$

Thus this improper integral diverges (to infinity).

Solutions:

4. $\displaystyle\int_0^1 \frac{1}{x^3}\, dx$

$$\begin{aligned}
\int_0^1 \frac{1}{x^3}\, dx &= \lim_{R\to 0^+} \int_R^1 x^{-3}\, dx \\
&= \lim_{R\to 0^+} -\frac{1}{2} x^{-2} \Big|_R^1 \\
&= \lim_{R\to 0^+} -\frac{1}{2} + \frac{1}{2R^2} \\
&= \infty
\end{aligned}$$

This improper integral also diverges (to infinity).