# Definition:

If $f$ is continuous on the interval $[a, b)$, but $|f(x)| \to \infty$ as $x \to b^-$, we define the **improper integral** of $f$ on $[a, b]$ by

$$\int_a^b f(x)\,dx = \lim_{R \to b^-} \int_a^R f(x)\,dx.$$

Similarly, if $f$ is continuous on the interval $(a, b]$, but $|f(x)| \to \infty$ as $x \to a^+$, we define the improper integral

$$\int_a^b f(x)\,dx = \lim_{R \to a^+} \int_R^b f(x)\,dx.$$

In either case, if the limit exists and equals some finite value $L$, we say that the improper integral **converges** to $L$. If the limit does not exist (whether because it equals $\pm\infty$ or because for some other reason), we say that the improper integral **diverges**.

# Important Point:

Just because $\lim_{x \to c} f(x) = \pm\infty$ does not mean that $\int_a^b f(x)\,dx$ will be automatically be infinite (if $c$ lies in the interval of integration).

**Examples:**

Both $\frac{1}{\sqrt{x}}$ and $\frac{1}{x^2} \to \infty$ as $x \to 0$.

BUT $\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$, while $\int_0^1 \frac{1}{x^2}\,dx = \infty$.

# In Class Work

1. Discuss the meaning of *convergent* and *divergent*. What does it mean for an *improper integral* to converge, as opposed to what it means for a *function* to converge?

2. For each integral: Why is it improper? Calculate, if it converges.

   (a) $\displaystyle\int_0^8 \frac{1}{x^{1/3}}\,dx$

   (b) $\displaystyle\int_{-8}^{-7} \frac{1}{(7+x)^3}\,dx$

   (c) $\displaystyle\int_0^{\pi^2} \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$

   (d) $\displaystyle\int_0^2 \frac{2x}{(x^2-1)^2}\,dx$

3. For what values of $p$ is $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ improper? For which of those values does the integral converge?

4. First determine why $\displaystyle\int_0^1 x\ln(x)\,dx$ is improper, and then (try to) evaluate it. What technique that most but not all of you learned in Calc 1 do you have to use?

# Solutions:

1. For each integral: Why is it improper? Calculate, if it converges.
   (a) $\int_0^8 \frac{1}{x^{1/3}}\, dx$
   The integrand $\frac{1}{x^{1/3}}$ is not continuous at $x = 0$, which is in the interval of integration.

$$\begin{aligned} \int_0^8 \frac{1}{x^{1/3}}\, dx &= \lim_{R\to 0^+} \int_R^8 x^{-1/3}\, dx \\ &= \lim_{R\to 0^+} \frac{3}{2} x^{2/3} \Big|_R^8 \\ &= \frac{3}{2} \lim_{R\to 0^+} 8^{2/3} - R^{2/3} \\ &= \frac{3}{2} \cdot 4 = 6 \end{aligned}$$

   Thus the improper integral converges, to 6.

# Solutions:

1(b) $\displaystyle\int_{-8}^{-7} \frac{1}{(7+x)^3}\, dx$

Improper because the integrand is not continuous on all of $[-8, -7]$.

$$\begin{aligned}
\int_{-8}^{-7} \frac{1}{(7+x)^3} &= \lim_{R\to -7^-} \int_{-8}^{R} \frac{1}{(7+x)^3}\, du \\
&\quad \text{Let } u = 7+x. \text{ Then } du = dx. \\
&= \lim_{R\to -7^-} \int_{-8}^{R} \frac{1}{u^3}\, du = \lim_{R\to -7^-} -\frac{1}{2}u^{-2}\Big|_{x=-8}^{x=R} \\
&= \lim_{R\to -7^-} -\frac{1}{2}\frac{1}{(7+x)^2}\Big|_{x=-8}^{x=R} \\
&= -\frac{1}{2}\left[\lim_{R\to -7^-}\left(\frac{1}{(7+R)^2} - \frac{1}{(-1)^2}\right)\right]
\end{aligned}$$

$\displaystyle\lim_{R\to -7^-} \frac{1}{(7+R)^2}$ diverges to $-\infty$, so this improper integral diverges.

## Solutions:

1(c) $\displaystyle\int_0^{\pi^2} \frac{\sin(\sqrt{x})}{\sqrt{x}}\, dx$

Improper because the integrand is not continuous on all of $[0, \pi^2]$.

$$\int_0^{\pi^2} \frac{\sin(\sqrt{x})}{\sqrt{x}}\, dx = \lim_{R\to 0^+} \int_R^{\pi^2} \frac{\sin(\sqrt{x})}{\sqrt{x}}\, dx$$

Let $u = \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}\, dx$

$$= \lim_{R\to 0^+} 2\int_{x=R}^{x=\pi^2} \sin(u)\, du = \lim_{R\to 0^+} -2\cos(u)\Big|_{x=R}^{x=\pi^2}$$

$$= \lim_{R\to 0^+} -2\cos(\sqrt{x})\Big|_{x=R}^{x=\pi^2}$$

$$= \lim_{R\to 0^+} -2\Big(\cos(\pi) - \cos(\sqrt{R})\Big) = -2(-1-1)$$

Thus this improper integral converges, to 4.

# Solutions:

1(d) $\displaystyle\int_0^2 \frac{2x}{(x^2-1)^2}\,dx$

Improper because the integrand is not continuous on all of $[0,2]$.

$$\begin{aligned}
\int_0^2 \frac{2x}{(x^2-1)^2}\,dx &= \int_0^1 \frac{2x}{(x^2-1)^2}\,dx + \int_1^2 \frac{2x}{(x^2-1)^2}\,dx \\
&= \lim_{S\to 1^-}\int_0^S \frac{2x}{(x^2-1)^2}\,dx + \lim_{R\to 1^+}\int_R^2 \frac{2x}{(x^2-1)^2}\,dx \\
&\qquad \text{Let } u = x^2-1. \text{ Then } du = 2x\,dx \\
&= \lim_{S\to 1^-}\int_{x=0}^{x=S} \frac{1}{u^2}\,du + \lim_{R\to 1^+}\int_{x=R}^{x=2} \frac{1}{u^2}\,du \\
&= \lim_{S\to 1^-} -\frac{1}{u}\Bigg|_{x=0}^{x=S} + \lim_{R\to 1^+} -\frac{1}{u}\Bigg|_{x=R}^{x=2} \\
&= \lim_{S\to 1^-}\left(-\frac{1}{S^2-1}+\frac{1}{-1}\right) + \lim_{R\to 1^+}\left(-\frac{1}{3}+\frac{1}{R^2-1}\right)
\end{aligned}$$

# Solutions:

1(d) $\displaystyle\int_0^2 \frac{2x}{(x^2-1)^2}\,dx$ *(Continued)*

$$\int_0^2 \frac{2x}{(x^2-1)^2}\,dx = \lim_{S\to 1^-}\left(-\frac{1}{S^2-1}+\frac{1}{-1}\right)+\lim_{R\to 1^+}\left(-\frac{1}{3}+\frac{1}{R^2-1}\right)$$

$$\lim_{S\to 1^{-1}} -\frac{1}{S^2-1} = -\infty \text{ and } \lim_{R\to 1^+}\frac{1}{R^2-1} = +\infty.$$

- Because $\displaystyle\int_0^2 \frac{2x}{(x^2-1)^2}\,dx =$ a sum with two terms that diverge, it diverges.
- $-\infty + \infty$ is not 0
- Notice that if we hadn't used the distinct variables $R$ and $S$, but instead had used $R$ for both, the two terms involving $R$ and $S$ would cancel, and we would (incorrectly) conclude that this improper integral converges.
- If either piece of a sum diverges, the whole sum diverges – two divergent pieces do not cancel.

# Solutions

2. *Due for Homework*

## Solutions:

3. $\displaystyle\int_0^1 x\ln(x)\,dx = \lim_{R\to 0^+}\int_R^1 x\ln(x)\,dx.$

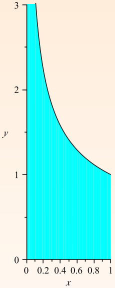
$u = \ln(x) \qquad dv = x\,dx$

$du = \frac{1}{x}\,dx \qquad v = \frac{1}{2}x^2$

$$\begin{aligned}
\int_0^1 x\ln(x)\,dx &= \lim_{R\to 0^+}\left[\frac{x^2}{2}\ln(x) - \int \frac{x^2}{2x}\,dx\right]_R^1 \\
&= \lim_{R\to 0^+}\left[\frac{x^2}{2}\ln(x) - \frac{1}{4}x^2\right]_R^1 = \lim_{R\to 0^+}\frac{x^2}{4}\Big[2\ln(x) - 1\Big]_R^1 \\
&= \lim_{R\to 0^+}\left[\frac{1}{4}(-1) - \frac{R^2}{4}\big(2\ln(R) - 1\big)\right] \\
&= -\frac{1}{4} - \lim_{R\to 0^+}\frac{R^2}{2}\big(\ln(R)\big) + \lim_{R\to 0^+}\frac{R^2}{4} \\
&= -\frac{1}{4} - \lim_{R\to 0^+}\frac{R^2}{2}\cdot\lim_{R\to 0^+}\ln(R) + 0 = -\frac{1}{4} - 0\cdot -\infty \\
&= ??
\end{aligned}$$

# Example:

Consider again the improper integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$. We've seen that this converges, to 2.




This "area" is 2, whether we find it by integrating with respect to $x$ or with respect to $y$! In other words ...

$$2 = \int_0^1 \frac{1}{\sqrt{x}}\,dx = \int_0^1 1\,dy + \int_1^\infty \frac{1}{y^2}\,dy.$$

## Definition:

If $f(x)$ is continuous on the interval $[a, \infty)$, we define the **improper integral** $\int_a^\infty f(x)\,dx$ to be

$$\int_a^\infty f(x)\,dx \stackrel{\text{def}}{=} \lim_{R\to\infty} \int_a^R f(x)\,dx.$$

Similarly, if $f(x)$ is continuous on the interval $(-\infty, a]$, we define

$$\int_{-\infty}^a f(x)\,dx \stackrel{\text{def}}{=} \lim_{R\to\infty} \int_{-R}^a f(x)\,dx.$$

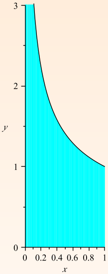
In either case, if the limit exists (and equals some value $L$), we say that the improper integral **converges** (to $L$). If the limit does not exist (whether because it is infinite or for other reasons), we say that the improper integral **diverges**.

1. Each of the following integrals is improper because the interval of integration is infinite. Determine whether each integral converges or diverges, and if it does converge, what it converges to.
   a. $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$
   b. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$
   c. $\displaystyle\int_1^\infty 1 + \frac{1}{x^2}\,dx$
   d. $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ where $p > 1$
2. Think about the above results and the big picture of what's going on.
   2.1 Is it *necessary* that $f(x)$ converge to 0 as $x \to \infty$ in order for $\displaystyle\int_a^\infty f(x)\,dx$ to converge to a finite number?
   2.2 If $f(x)$ does converge to 0 as $x \to \infty$, *must* $\displaystyle\int_a^b f(x)\,dx$ automatically converge to a finite number? That is, is $f(x) \to 0$ a *sufficient* condition for $\displaystyle\int_a^\infty f(x)\,dx$ to converge to a finite number?

# Recall

- We've seen that the improper integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges, to 2.

  We can think of the signed area of the region below as being 2.



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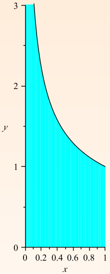
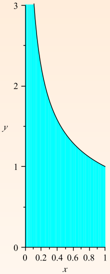



- This signed area is 2 (in the limit) no matter how we find it: that is, whether we integrate w.r.t. to $x$ or w.r.t. $y$! In other words ...

# Recall

- We've seen that the improper integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges, to 2. We can think of the signed area of the region below as being 2.




- This signed area is 2 (in the limit) no matter how we find it: that is, whether we integrate w.r.t. to $x$ or w.r.t. $y$! In other words ...
- $$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2 = \int_0^1 1\,dy + \int_1^\infty \frac{1}{y^2}\,dy.$$