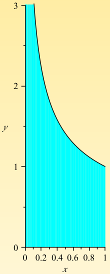
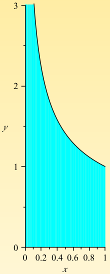
Recall

- We've seen that the improper integral $\int_0^1 \frac{1}{\sqrt{x}}\, dx$ converges, to 2.
  We can think of the signed area of the region below as being 2.



Recall

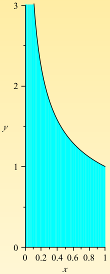
- We've seen that the improper integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges, to 2. We can think of the signed area of the region below as being 2.



- This signed area is 2 (in the limit) no matter how we find it: that is, whether we integrate w.r.t. to x or w.r.t. y! In other words ...

Recall

- We've seen that the improper integral $\int_0^1 \frac{1}{\sqrt{x}}\, dx$ converges, to 2. We can think of the signed area of the region below as being 2.



- This signed area is 2 (in the limit) no matter how we find it: that is, whether we integrate w.r.t. to x or w.r.t. y! In other words ...
- $$\int_0^1 \frac{1}{\sqrt{x}}\, dx = 2 = \int_0^1 1\, dy + \int_1^\infty \frac{1}{y^2}\, dy.$$

Definition:

If $f(x)$ is continuous on the interval $[a, \infty)$, we define the **improper integral** $\int_a^\infty f(x)\, dx$ to be

$$\int_a^\infty f(x)\, dx \stackrel{\text{def}}{=} \lim_{R \to \infty} \int_a^R f(x)\, dx.$$

Similarly, if $f(x)$ is continuous on the interval $(-\infty, a]$, we define

$$\int_{-\infty}^a f(x)\, dx \stackrel{\text{def}}{=} \lim_{R \to \infty} \int_{-R}^a f(x)\, dx.$$

In either case, if the limit exists (and equals some value L), we say that the improper integral **converges** (to L). If the limit does not exist (whether because it is infinite or for other reasons), we say that the improper integral **diverges**.

In Class Work

1. As $x \to \infty$, does each *integrand* diverge or converge (if so, to what?) Also, does each improper *integral* diverge or converge (if so, to what?)

   a. $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$ b. $\displaystyle\int_1^\infty 1 + \frac{1}{x^2}\,dx$ c. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ d. $\displaystyle\int_0^\infty xe^{-x^2}\,dx$

2. Think about all the results you've seen, as well as the big picture.

   (a) Is it *necessary* that $f(x)$ converge to 0 as $x \to \infty$ in order for $\displaystyle\int_a^\infty f$ to converge to a finite number?

   (b) If $f(x)$ *does* converge to 0 as $x \to \infty$, *must* $\displaystyle\int_a^\infty f$ converge to a finite number? That is, is $f(x) \to 0$ a *sufficient* condition for $\displaystyle\int_a^\infty f$ to converge to a finite number?

Important Lessons:

1. There is a huge distinction between $f(x)$ converging – that is, $\lim_{x\to\infty} f(x)$ being finite – and $\int_a^\infty f(x)\,dx$ converging. Just because you can find $\lim_{x\to\infty} f(x)$, and it's a finite number, does **not** mean that $\int_a^\infty f(x)\,dx$ will be finite.

Important Lessons:

1. There is a huge distinction between $f(x)$ converging – that is, $\lim_{x\to\infty} f(x)$ being finite – and $\int_a^\infty f(x)\,dx$ converging. Just because you can find $\lim_{x\to\infty} f(x)$, and it's a finite number, does **not** mean that $\int_a^\infty f(x)\,dx$ will be finite.
2. In fact, if $\lim_{x\to\infty} f(x)$ exists **but is not 0**, $\int_a^\infty f$ diverges! No need to investigate any further.

Important Lessons:

1. There is a huge distinction between $f(x)$ converging – that is, $\lim_{x\to\infty} f(x)$ being finite – and $\int_a^\infty f(x)\,dx$ converging. Just because you can find $\lim_{x\to\infty} f(x)$, and it's a finite number, does **not** mean that $\int_a^\infty f(x)\,dx$ will be finite.
2. In fact, if $\lim_{x\to\infty} f(x)$ exists **but is not 0**, $\int_a^\infty f$ diverges! No need to investigate any further.
3. If $\lim_{x\to\infty} f(x)$ **is** 0, $\int_a^\infty f$ may converge or it may diverge – to find out, you must actually do the antidifferentiation and the limit.

Solutions

1(a) $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$

- $1/x^3$ converges to 0 as $x \to \infty$
- $$\begin{aligned}\int_1^\infty \frac{1}{x^3}\,dx &= \lim_{R\to\infty}\int_1^R x^{-3}\,dx = \lim_{R\to\infty}\left(-\frac{1}{2}\cdot\frac{1}{x^2}\right)\Big|_1^R \\ &= \lim_{R\to\infty}\left(-\frac{1}{2R^2}+\frac{1}{2}\right) = \frac{1}{2}\end{aligned}$$

  This improper integral converges, to $\frac{1}{2}$.
- **Results:** Integrand converges to 0; integral converges.

Solutions

1(b) $\displaystyle\int_1^\infty 1 + \frac{1}{x^2}\ dx$

- $1 + \frac{1}{x^2} \to 1$ as $x \to \infty$
- $$\int_1^\infty 1 + \frac{1}{x^2}\ dx = \int_1^\infty 1\ dx + \int_1^\infty \frac{1}{x^2}\ dx$$

  We've seen that the first integral on the right diverges (to ∞), the second one converges (to 1).

  Because this sum does not approach a finite number, it *diverges*.
- **Result:** Integrand converges to 1; integral diverges.

Solutions

1(c) $\displaystyle\int_1^\infty \frac{1}{x}\, dx$

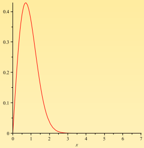
- $\dfrac{1}{x} \to 0$ as $x \to \infty$
- $$\begin{aligned}\int_1^\infty \frac{1}{x}\, dx &= \lim_{R\to\infty} \int_1^R \frac{1}{x}\, dx = \lim_{R\to\infty} \ln(x)\Big|_1^R \\ &= \lim_{R\to\infty} (\ln(R) - \ln(1)) = \lim_{R\to\infty} \ln(R) = \infty\end{aligned}$$

  This improper integral diverges (slowly).
- **Result:** Integrand converges to 0; integral diverges.

Solutions

1(d) $\displaystyle\int_0^\infty xe^{-x^2}\,dx$

- As $x \to \infty$, $\frac{x}{e^{x^2}} \to \frac{\infty}{\infty}$. Another limit we can't do b/c it's in *indeterminate form*! Looking at a graph of xe^{-x^2}, can see that integrand approaches 0.



- Let $u = -x^2$, so $du = -2x\,dx$, or $-\frac{1}{2}\,du = x\,dx$
  Also, $x = 0 \Rightarrow u = 0$; $x = \infty \Rightarrow u = -\infty$.

$$\begin{aligned}\int_0^\infty xe^{-x^2}\,dx &= -\frac{1}{2}\int_0^{-\infty} e^u\,du = \frac{1}{2}\int_{-\infty}^0 e^u\,du = \frac{1}{2}\lim_{R\to\infty} e^u\Big|_{-R}^0 \\ &= \frac{1}{2}\lim_{R\to\infty} 1 - e^{-R} = \frac{1}{2}\left(1 - \lim_{R\to\infty}\frac{1}{e^R}\right) = \frac{1}{2}\end{aligned}$$

  The improper integral converges, to 1.
- **Result:** Integrand converges to 0; integral converges.

Solutions

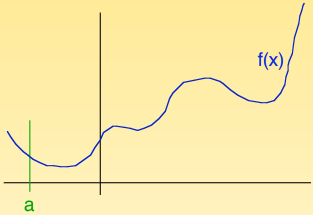
2. Think about all the results you've seen, as well as the big picture.
   (a) Is it *necessary* that $f(x)$ converge to 0 as $x \to \infty$ in order for $\int_a^\infty f(x)\,dx$ to converge to a finite number?

Convergent integral	What f converges to
$\int_1^\infty \frac{1}{x^2}\,dx$	$\frac{1}{x^2} \to 0$
$\int_1^\infty \frac{1}{x^3}\,dx$	$\frac{1}{x^3} \to 0$
$\int_0^\infty xe^{-x^2}\,dx$	$xe^{-x^2} \to 0$

For what it's worth, so far every example that we've seen of a convergent improper integral *has* had an integrand that converges to 0 as $x \to \infty$. But that's not enough.

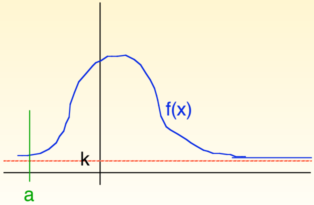
Solutions:

2(a) (continued)



As $x \to \infty$, $f(x) \to \infty$,

and $\displaystyle\int_a^\infty f(x)\,dx = \infty$

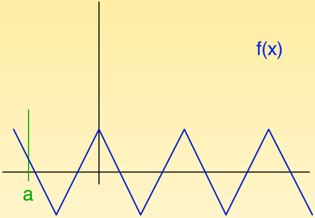


As $x \to \infty$, $f(x) \to k \neq 0$, and

$$\begin{aligned}\int_a^\infty f(x)\,dx &= \lim_{R\to\infty}\int_a^R f(x)\,dx\\ &> \lim_{R\to\infty}\int_a^R k\,dx\\ &> \lim_{R\to\infty} kR = \pm\infty\end{aligned}$$

Solutions:

2(a) (continued)



As $x \to \infty$, $\lim\limits_{x\to\infty} f(x)$ d.n.e., and $\displaystyle\int_a^\infty f(x)\,dx = \lim_{R\to\infty} f(x)\,dx$ d.ne., so the integral diverges

Conclusion: The only way $\displaystyle\int_a^\infty f(x)\,dx$ can have a **hope** of converging to a finite number is if $\lim_{x\to\infty} f(x) = 0$.

In other words, if $\lim_{x\to\infty} f(x) \neq 0$, $\displaystyle\int_a^\infty f(x)\,dx$ **must** diverge.

Solutions:

2(b) If $f(x)$ does converge to 0 as $x \to \infty$, *must* $\int_a^b f(x)\,dx$ automatically converge to a finite number? That is, is $f(x) \to 0$ a *sufficient* condition for $\int_a^\infty f(x)\,dx$ to converge to a finite number?

Functions that converge to 0	What $\int_a^\infty f(x)\,dx$ does
xe^{-x^2}	converges
$\frac{1}{x^2}$	converges
$\frac{1}{x^3}$	converges
$\frac{1}{x}$	diverges

Thus knowing that $f(x) \to \infty$ is *not* sufficient information to conclude that $\displaystyle\int_a^\infty f(x)\,dx$ converges!