Recall

- A function $f(x)$ converges as $x \to \infty$ if $\lim\limits_{x\to\infty} f(x)$ is finite, i.e. if $f(x)$ has a horizontal asymptote on the right side of the graph.
- An improper integral $\int_a^\infty f(x)\,dx$ converges if $\lim_{R\to\infty} \int_a^R f(x)\,dx$ is finite, i.e. if the signed area between f and the x-axis is finite.
- There is a big distinction between $f(x)$ converging and $\int_a^\infty f(x)\,dx$ converging. $\lim\limits_{x\to\infty} f(x)$ being finite does **not** guarantee that $\int_a^\infty f(x)\,dx$ will be finite.

Recall: Examples of Infinite Intervals of Integration

Problem		As $x \to \infty$,	$\int_1^\infty f(x)\,dx$
	$f(x) = 2$	$f(x)$ converges (to 2)	$\int_1^\infty f(x)\,dx$ diverges (to ∞)
	$f(x) = 10^{-100}$	$f(x)$ converges (to 10^{-100})	$\int_1^\infty f(x)\,dx$ diverges (to ∞)
	$f(x) = \frac{1}{x^2}$	$f(x)$ converges (to 0)	$\int_1^\infty f(x)\,dx$ converges (to 2)
1(a)	$f(x) = \frac{1}{x^3}$	$f(x)$ converges (to 0)	$\int_1^\infty f(x)\,dx$ converges (to $\frac{1}{2}$)
1(b)	$f(x) = 1 + \frac{1}{x^2}$	$f(x)$ converges (to 1)	$\int_1^\infty f(x)\,dx$ diverges (to ∞)
1(c)	$f(x) = \frac{1}{x}$	$f(x)$ converges (to 0)	$\int_1^\infty f(x)\,dx$ diverges (to ∞)
1(d)	$f(x) = xe^{-x^2}$	$f(x)$ converges (to 0)	$\int_1^\infty f(x)\,dx$ converges (to $\frac{1}{2}$)

In Class Work From Thursday

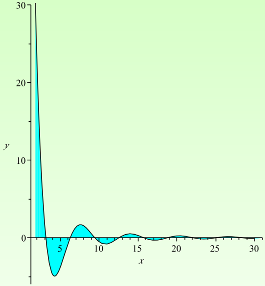
2. Think about all the results you've seen, as well as the big picture.
   (a) Is it *necessary* that $f(x)$ converge to 0 as $x \to \infty$ in order for $\displaystyle\int_a^\infty f$ to converge to a finite number?
   (b) If $f(x)$ *does* converge to 0 as $x \to \infty$, *must* $\displaystyle\int_a^\infty f$ converge to a finite number? That is, is $f(x) \to 0$ a *sufficient* condition for $\displaystyle\int_a^\infty f$ to converge to a finite number?

Important Conclusion:

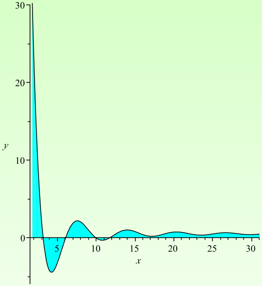
- A function $f(x)$ converges as $x \to \infty$ if $\lim_{x\to\infty} f(x)$ is finite, i.e. if $f(x)$ has a horizontal asymptote on the right side of the graph.
- An improper integral $\int_a^\infty f(x)\,dx$ converges if $\lim_{R\to\infty} \int_a^R f(x)\,dx$ is finite, i.e. if the signed area between f and the x-axis is finite.
- There is a big distinction between $f(x)$ converging and $\int_a^\infty f(x)\,dx$ converging. $\lim_{x\to\infty} f(x)$ being finite does **not** guarantee that $\int_a^\infty f(x)\,dx$ will be finite.
- In fact, if $\lim_{x\to\infty} f(x)$ exists but is not 0, $\int_a^\infty f(x)\,dx$ diverges! (If $f(x)$ itself diverges, $\int_a^\infty f(x)\,dx$ will of course diverge.)
- If $\lim_{x\to\infty} f(x) = 0$, $\int_a^b f(x)\,dx$ may converge or it may diverge – you must investigate further.

Infinite Intervals of Integration

- The only way $\int_a^{\infty} f(x)\,dx$ can have a **hope** of converging to a finite number is if the integrand, $f(x)$, converges to 0 as $x \to \infty$



$\lim\limits_{x\to\infty} f = 0$ (f converges to 0)
$\int_a^{\infty} f$ may or may not converge to a finite signed area



$\lim\limits_{x\to\infty} f = \frac{1}{2}$ (f converges to $1/2$)
$\int_a^{\infty} f$ can **not** converge to a finite signed area

In Class Work

1. Let $I = \int_1^\infty \frac{1}{x^p}\, dx$. Determine whether I converges or diverges if
   (a) $p > 1$
   (b) $p = 1$
   (c) $0 < p < 1$
   (d) $p = 0$
   (e) $p < 0$

Solutions:

1. Let $I = \int_1^\infty \frac{1}{x^p}\,dx$. Determine whether I converges or diverges if
   (a) $p > 1$:

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{R\to\infty}\int_1^R x^{-p}\,dx = \lim_{R\to\infty}\frac{1}{-p+1}x^{-p+1}\Big|_1^R \\ &= \lim_{R\to\infty}\frac{1}{1-p}R^{1-p} - \frac{1}{1-p}.\end{aligned}$$

$p > 1 \Rightarrow 1-p < 0 \Rightarrow p-1 > 0$

$$\lim_{R\to\infty} R^{1-p} = \lim_{R\to\infty}\frac{1}{R^{p-1}} = 0.$$

Thus $\int_1^\infty \frac{1}{x^p}\,dx = 0 - \frac{1}{1-p} = \frac{1}{p-1}$.

Solutions:

1(b) $p = 1$

We've already done this case. We know that $\displaystyle\int_1^\infty \frac{1}{x}\,dx = \infty$.

1(c) $0 < p < 1$:

As in the case of $p > 1$,

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{R\to\infty} \frac{1}{1-p}R^{1-p} - \frac{1}{1-p}.$$

$0 < p < 1 \Rightarrow 1 - p > 0 \Rightarrow \lim\limits_{R\to\infty} R^{1-p} \to \infty$.

In this case, the integral again diverges.

Solutions:

1(d) $p = 0$:

$$\int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty 1\,dx = \infty,$$

as we've already seen.

1(e) $p < 0$:
Let $a = -p > 0$.

$$\int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty x^{-p}\,dx = \int_1^\infty x^a\,dx.$$

As $x \to \infty$, $x^a \to \infty$, and so the area under this curve is clearly infinite!