Recall:

$$D_{\overrightarrow{\mathbf{u}}} f(a,b) \stackrel{\text{def}}{=} < f_x(a,b), f_y(a,b) > \cdot \overrightarrow{\mathbf{u}}$$
$$= \nabla f(a,b) \cdot \overrightarrow{\mathbf{u}}$$

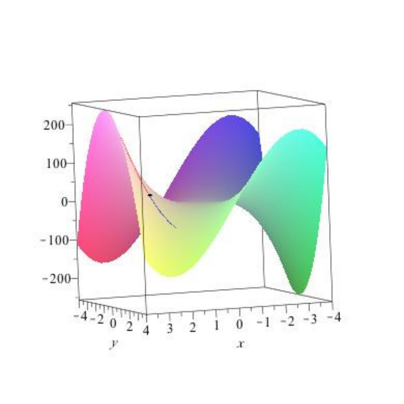
So when is the gradient vector at a point perpendicular to a unit vector $\overrightarrow{\mathbf{u}}$?

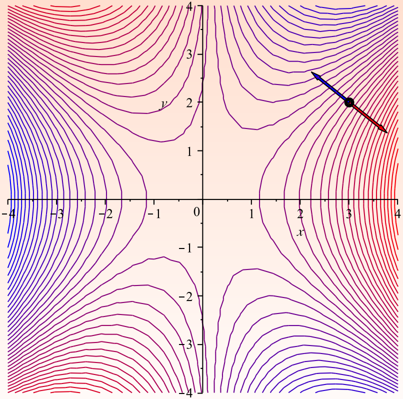
$$\begin{aligned} \nabla f(a,b) \perp \overrightarrow{\mathbf{u}} \quad &\Leftrightarrow \quad D_{\overrightarrow{\mathbf{u}}} f(a,b) = 0 \\ &\Leftrightarrow \quad f \text{ isn't changing in the direction of } \overrightarrow{\mathbf{u}} \\ &\Leftrightarrow \quad \overrightarrow{\mathbf{u}} \text{ is tangent to the level curve through } f(a,b). \end{aligned}$$

Recall

We looked at the surface $f(x, y) = 4x^3 - 6xy^2 + y^2$.
$\nabla f(3, 2)$ is shown in red and $-\nabla f(3, 2)$ is shown in blue on both the surface and the contour plot.

Notice that these look orthogonal to the level curve at at $(3, 2)$.

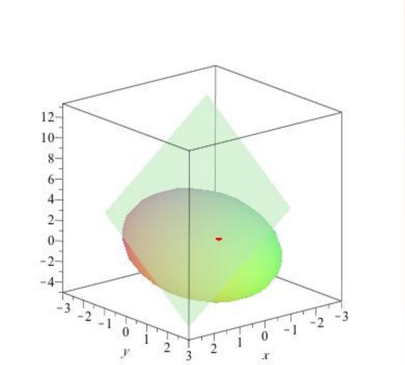


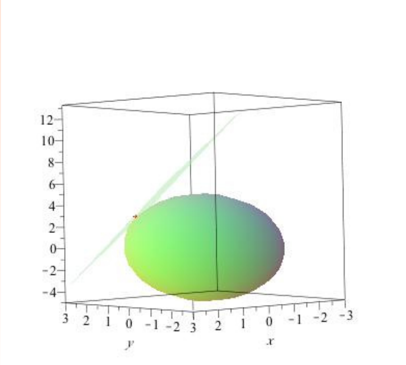


Also recall:

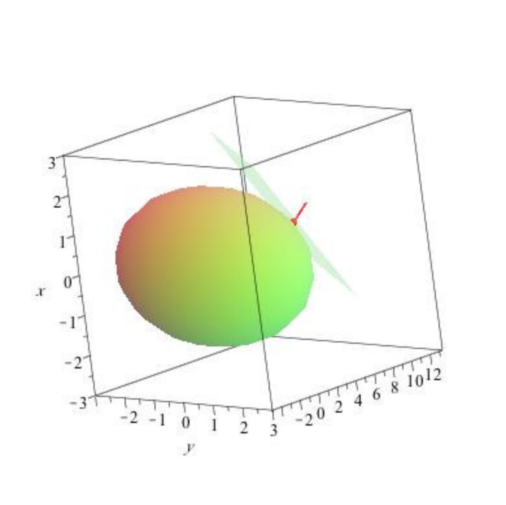
Last Friday, we found the equation of the plane tangent to the ellipsoid $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{25}=1$ at the point $\left(1,2,5\sqrt{11}/6\right)$, using implicit differentiation. We found that at the point $(1,2,\frac{5\sqrt{11}}{6})$, the equation of the tangent plane is

$$0=-\frac{15}{2\sqrt{11}}(x-1)-\frac{20}{3\sqrt{11}}(y-2)-(z-\frac{5\sqrt{11}}{6}).$$





After a LOT of fiddling around, I was able to find an angle where you almost believe that the gradient vector (the red line) is indeed orthogonal to the tangent plane:



Check algebraically that the gradient is indeed normal to the tangent plane:

Tangent plane at $(1, 2, \frac{5\sqrt{11}}{6})$:

$-\frac{15}{2\sqrt{11}}(x-1) - \frac{20}{3\sqrt{11}}(y-2) - 1(z - \frac{5\sqrt{11}}{6}) = 0$

$\Longrightarrow$ The normal vector is $\overrightarrow{\mathbf{n}} = \left\langle -\frac{15}{2\sqrt{22}}, -\frac{20}{3\sqrt{11}}, -1 \right\rangle$

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Gradient vector at $(1, 2, \frac{5\sqrt{11}}{6})$:

Thinking of the ellipsoid as a level surface of $f(x,y,z) = \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} - 1$,

$$\nabla f = < \frac{x}{2}, \frac{2y}{9}, \frac{2z}{25} > \quad \Longrightarrow \quad \nabla f\left(1, 2, \frac{5\sqrt{11}}{6}\right) = \left\langle \frac{1}{2}, \frac{4}{9}, \frac{\sqrt{11}}{15} \right\rangle.$$

Check algebraically that the gradient is indeed normal to the tangent plane:

Tangent plane at $(1, 2, \frac{5\sqrt{11}}{6})$:

$-\frac{15}{2\sqrt{11}}(x-1) - \frac{20}{3\sqrt{11}}(y-2) - 1(z - \frac{5\sqrt{11}}{6}) = 0$

$\implies$ The normal vector is $\overrightarrow{\mathbf{n}} = \left\langle -\frac{15}{2\sqrt{22}}, -\frac{20}{3\sqrt{11}}, -1 \right\rangle$

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Thinking of the ellipsoid as a level surface of $f(x, y, z) = \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} - 1$,

$$\nabla f = < \frac{x}{2}, \frac{2y}{9}, \frac{2z}{25} > \qquad \implies \qquad \nabla f\left(1, 2, \frac{5\sqrt{11}}{6}\right) = \left\langle \frac{1}{2}, \frac{4}{9}, \frac{\sqrt{11}}{15} \right\rangle.$$

With some checking you can see that

$$\overrightarrow{\mathbf{n}} = -\frac{15}{\sqrt{11}} \nabla f\left(1, 2, \frac{5\sqrt{11}}{6}\right).$$

$$\vec{\mathbf{n}} = -\frac{15}{\sqrt{11}} \nabla f \left(1, 2, \frac{5\sqrt{11}}{6}\right)$$

Thus the gradient vector and the normal vector to the tangent plane are parallel.

In other words,

the gradient at $\left(1, 2, \frac{5\sqrt{11}}{6}\right)$ is normal to the tangent to the level surface $0 = f(x, y, z)$ at the point $\left(1, 2, \frac{5\sqrt{11}}{6}\right)$, as claimed.

1. Find all points at which the tangent plane to $z = 2x^2 - 4xy + y^4$ is parallel to the xy-plane. Discuss the graphical significance of each point.