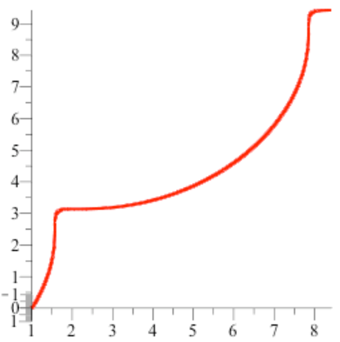
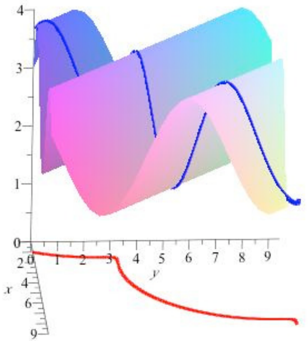
Integrating over a curve $\mathcal{C}$

Suppose we have a curve $\mathcal{C}$ in $\mathbb{R}^2$ and a surface $z = f(x, y)$.





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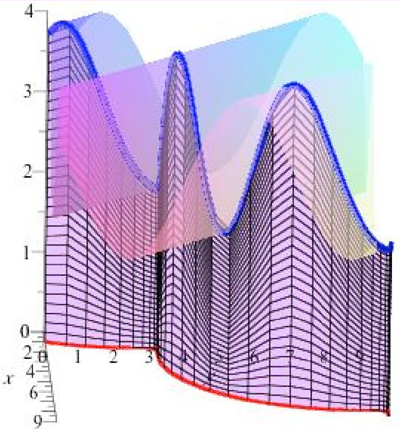
We want to find the area of the vertical surface that we create if we go straight up from the curve $\mathcal{C}$ to the surface.



Quick Arclength Reminder

$$
\begin{aligned}
s(t) &= \int_{t_0}^{t} \sqrt{[x'(u)]^2 + [y'(u)]^2}\, du \\
&= \int_{t_0}^{t} \|\overrightarrow{\mathbf{r}}\,'(u)\|\, du \\
\Longrightarrow \frac{ds}{dt} &= \|\overrightarrow{\mathbf{r}}\,'(t)\| \\
\Longrightarrow ds &= \|\overrightarrow{\mathbf{r}}\,'(t)\|\, dt
\end{aligned}
$$

In Class Work

1. Evaluate the line integral with respect to arclength $\int_{\mathcal{C}} 2x \, ds$, where $\mathcal{C}$ is the line segment from $(1, 2)$ to $(3, 5)$ by doing the following:

   (a) Find the parametric equations $\overrightarrow{\mathbf{r}}(t) =< x(t), y(t) >$ for the line segment $\mathcal{C}$.

   Remember: For a line, $x(t) = x_0 + v_x t$, $y(t) = y_0 + v_y t$, where (x_0, y_0) is a point on the line and $< v_x, v_y >$ is a direction vector for the line.

   (b) Use the formula $\int_{\mathcal{C}} 2x \, ds = \int_a^b f\left(\overrightarrow{\mathbf{r}}(t)\right) \|\overrightarrow{\mathbf{r}}'(t)\| \, dt$.

2. Evaluate the line integral $\int_{\mathcal{C}} (3x - y) \, ds$ where $\mathcal{C}$ is the quarter-circle $x^2 + y^2 = 9$ from $(0, 3)$ to $(3, 0)$.

   Hint: As in #1, find parametric equations for $\mathcal{C}$ first.

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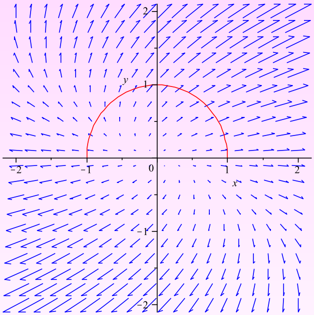
- What if we want to move an object in 3 dimensions? Or if the force isn't always pushing in the same direction as the object is moving?

  That is, what if an object is moving through a force field?

Work

A particle moves through the force field $\overrightarrow{\mathbf{F}}(x, y) = < x + y, y >$ along the curve $\mathcal{C}$ that corresponds to the upper unit semicircle oriented counter-clockwise.



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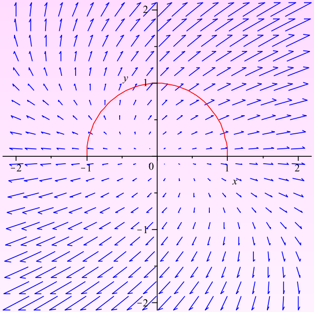

Question: How do we measure the work done *by the force field* in moving this particle along that curve? (In other words, how do we measure the contribution of the force field in getting the particle to where it ends up?)

More Arclength Reminders

Suppose we begin with a curve $\mathcal{C}$ already parametrized in terms of arclength, $\overrightarrow{\mathbf{r}}(s) =< x(s), y(s) >$.
While it may seem odd to do, we can still calculate arclength

$$s = \int_{s_0}^{s} \sqrt{[x'(u)]^2 + y'(u)]^2}\, du = \int_{s_0}^{s} \left\| \frac{d\overrightarrow{\mathbf{r}}}{du} \right\| du.$$

Differentiating both sides with respect to the arclength parameter s, we get

$$1 = \left\| \frac{d\overrightarrow{\mathbf{r}}}{ds} \right\|.$$

Thus

$$\overrightarrow{\mathbf{T}}(s) = \frac{d\overrightarrow{\mathbf{r}}}{ds}, \text{ since } \frac{d\overrightarrow{\mathbf{r}}}{ds} \text{ has length 1.}$$

We can use this to get

$$\boxed{d\overrightarrow{\mathbf{r}} = \overrightarrow{\mathbf{T}}\, ds}$$