Let $\overrightarrow{\mathbf{F}}(x, y) =< M(x, y), N(x, y) >=< y, -x >$ and let $\mathcal{C}$ be the unit circle oriented counterclockwise and let R be the region enclosed by $\mathcal{C}$. Calculate the following.

1. $\displaystyle\oint_{\mathcal{C}} \overrightarrow{\mathbf{F}}(x, y) \cdot d\overrightarrow{\mathbf{r}}$

Is $\overrightarrow{\mathbf{F}}$ conservative?

$$\frac{\partial}{\partial y}(y) = 1 \qquad \frac{\partial}{\partial x}(-x) = -1$$

Since these aren't equal, $\overrightarrow{\mathbf{F}}$ is not conservative, so we have to parametrize the curve:

$\mathcal{C} : \overrightarrow{\mathbf{r}}(t) =< \cos(t), \sin(t) >$, $0 \leq t \leq 2\pi$.

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$$\begin{aligned}
\oint_{\mathcal{C}} \overrightarrow{\mathbf{F}}(x,y) \cdot d\overrightarrow{\mathbf{r}} &= \int_0^{2\pi} < \sin(t), -\cos(t) > \cdot < -\sin(t), \cos(t) > \ dt \\
&= \int_0^{2\pi} -\sin^2(t) - \cos^2(t) \ dt \\
&= \int_0^{2\pi} -1 \ dt \\
&= -t \Big|_0^{2\pi} \\
&= -2\pi
\end{aligned}$$

A. Let $\overrightarrow{\mathbf{F}}(x, y) =< M(x, y), N(x, y) >=< y, -x >$ and let $\mathcal{C}$ be the unit circle oriented counterclockwise and let R be the region enclosed by $\mathcal{C}$. Calculate the following.

2. $\displaystyle\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA$

One approach:

$\mathcal{C}$ is a simple closed curve.

Counter-clockwise is positive orientation.

y and $-x$ are continuous and have continuous partials.

Thus Green's theorem applies and tells us that

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA = \oint_{\mathcal{C}} \overrightarrow{\mathbf{F}}(x, y) \cdot d\overrightarrow{\mathbf{r}} = -2\pi.$$

A. Let $\overrightarrow{\mathbf{F}}(x,y) = < M(x,y), N(x,y) > = < y, -x >$ and let $\mathcal{C}$ be the unit circle oriented counterclockwise and let R be the region enclosed by $\mathcal{C}$. Calculate the following.

2. $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA$

Another approach:

To check whether (1) whether Green's theorem is true or (2) whether Green's theorem gives us a substantially easier way to evaluate the double integral, do this double integral in the usual way.

$$\begin{aligned}
\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA &= \iint_R (-1 - 1) \, dA \\
&= -2 \iint_R \, dA \\
&= -2(\text{the area of the circle}) \\
&= -2\pi
\end{aligned}$$

B. Use Green's Theorem to evaluate $\oint_{\mathcal{C}} \overrightarrow{\mathbf{F}}(x, y) \cdot d\overrightarrow{\mathbf{r}}$ in each case. (Assume the orientation is positive).

1. $\overrightarrow{\mathbf{F}}(x, y) =< y^2 + x^2, x + y >$
$\mathcal{C}$ is the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$

Conservative?

$$\frac{\partial}{\partial x}(x + y) = 1 \qquad \frac{\partial}{\partial y}(y^2 + x^2) = 2y.$$

Since they're not equal, $\overrightarrow{\mathbf{F}}$ is not conservative.

Does Green's Theorem apply? $\mathcal{C}$ is a simple closed curve; the orientation is positive, and both $y^2 + x^2$ and $x + y$ are continuous and have continuous partials, so yes.

B. Use Green's Theorem to evaluate $\oint_{\mathcal{C}} \overrightarrow{\mathbf{F}}(x,y)\cdot d\overrightarrow{\mathbf{r}}$ in each case. (Assume the orientation is positive).

1. $\overrightarrow{\mathbf{F}}(x,y) = < y^2 + x^2, x+y >$

$\mathcal{C}$ is the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$

By Green's Theorem,

$$\begin{aligned}
\oint_{\mathcal{C}} \overrightarrow{\mathbf{F}}(x,y)\cdot d\overrightarrow{\mathbf{r}} &= \iint_R \frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial y}(y^2+x^2)\ dA \\
&= \iint_R 1-2y\ dA \\
&= \int_0^1\int_0^1 1-2y\ dx\ dy \\
&= \int_0^1 x-2yx\Big|_0^1\ dy \\
&= \int_0^1 (1-2y)\ dy = y-y^2\Big|_0^1 \\
&= 0
\end{aligned}$$

B. Use Green's Theorem to evaluate $\oint_{\mathcal{C}} \overrightarrow{\mathbf{F}}(x,y) \cdot d\,\overrightarrow{\mathbf{r}}$ in each case. (Assume the orientation is positive).

2. $\overrightarrow{\mathbf{F}}(x,y) = < x-y, x+y >$
$\mathcal{C}$ is the circle of radius 3 with center $(1,0)$

Conservative? $\frac{\partial}{\partial x}(x+y) = 1$, $\frac{\partial}{\partial y}(x-y) = -1$, so no.

$\mathcal{C}$ is a simple closed curve; the orientation is positive, and both $x-y$ and $x+y$ are continuous with continuous partials, so again, yes.

$$\begin{aligned}
\oint_{\mathcal{C}} \overrightarrow{\mathbf{F}}(x,y) \cdot d\,\overrightarrow{\mathbf{r}} &= \iint_R \frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial y}(x-y)\, dA \\
&= \iint_R 1-(-1)\, dA \\
&= 2(\text{area of a circle of radius 3}) \\
&= 2\pi(9) = 18\pi
\end{aligned}$$