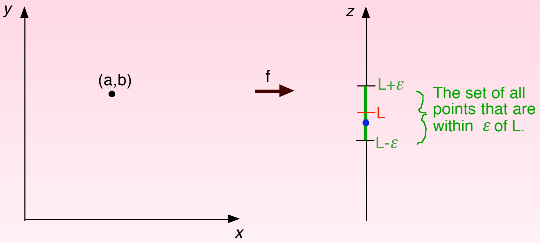
Idea behind the definition of the multivariate limit:

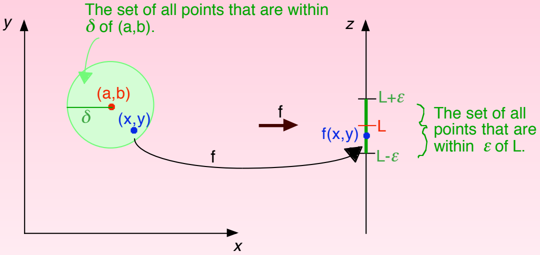
Given an element $(a, b) \in \mathbb{R}^2$, and a real number $L \in \mathbb{R}$. Does $\lim_{(x,y)\to(a,b)} f(x, y) = L$?

Let ϵ be an arbitrary positive real number. Mark off the set of all points within ϵ of L on the z-axis.



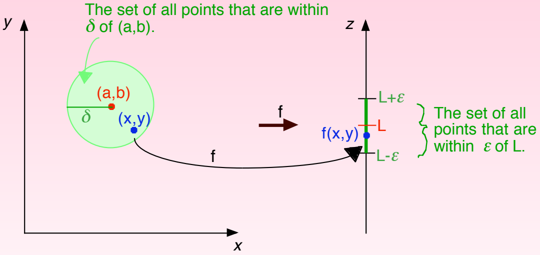
Idea behind the definition of the multivariate limit:

Does there exist a circle centered at (a, b), such that every point in that circle gets sent by f to the region around L that we marked off?



Idea behind the definition of the multivariate limit:

If, for *every* $\epsilon > 0$, there is a circle around (a, b) so that every point in the circle gets sent by f to the interval $(L - \epsilon, L + \epsilon)$, then we say that $\lim_{(x,y)\to(a,b)} f(x, y) = L$, because we can get arbitrarily close to L, just by choosing a small enough radius around (a, b).



Definition:

$\lim_{(x,y)\to(a,b)} f(x,y) = L$ if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d\left((x,y),(a,b)\right) < \delta \qquad \Longrightarrow \qquad |f(x,y) - L| < \epsilon$$

Example:

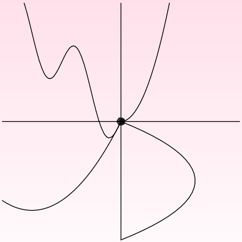
Let $z = f(x, y)$, where $f(x, y) = 1 - x^2 - y^2$. Because of your Calc 1 experience with limits, you will not be surprised that

$$\lim_{(x,y)\to(0,0)} f(x, y) = 1.$$

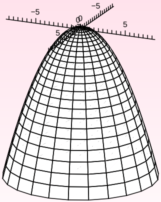
What that means:

No matter what path you look at that leads to $(0, 0)$, on the surface, that path approaches $z = 1$.

A few possible paths



The surface $z = 1 - x^2 - y^2$



Let $f(x,y) = \dfrac{6(x-1)^3(y-2)}{2(x-1)^4 + (y-2)^4}$.

Consider $\lim\limits_{(x,y)\to(1,2)} f(x,y)$.

Remember: all paths we try must go through the point $(1,2)$!

1. Find the limit as you approach the origin along the line $x = 1$.

2. Find the limit as you approach the origin along the line $y = 2$.

3. Find the limit as you approach the origin along the line $y = x + 1$.
   (*Notice that the point* $(1,2)$ *lies on this line.*)

4. Find the limit as you approach the origin along the line
   $y = m(x-1) + 2$.
   (*Again, notice that the point* $(1,2)$ *lies on this line.*)

5. What can you conclude about $\lim\limits_{(x,y)\to(1,2)} f(x,y)$?