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When does a tangent plane at (a, b) exist?

- In order for a tangent **line** to exist, all we need is for $f'(a)$ to exist. If it does, we say f is *differentiable* at $x = a$.

- Unfortunately, just because you can find the equation of the above plane does NOT automatically make it tangent to the surface!

  As we've discussed, in the two dimensional case, there are an infinite number of tangent lines at a point. In order for the concept of tangent plane to make sense, we need *all* of those tangent lines to lie in the same plane.

  A function that meets that condition will be **differentiable**.

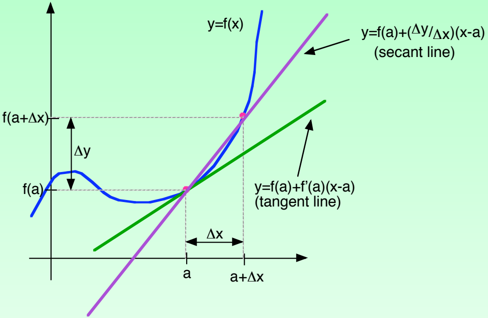
One definition of $f'(a)$:

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

If we let $\epsilon = \dfrac{\Delta y}{\Delta x} - f'(a)$, then $\epsilon \to 0$ as $\Delta x \to 0$, and

$$\Delta y = f'(a)\Delta x + \epsilon \Delta x.$$

Graphically:

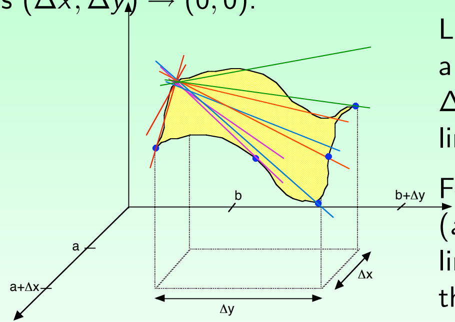


Let $\epsilon = \dfrac{\Delta y}{\Delta x} - f'(a) =$ the difference in the slope of the secant line on $[a, a+\Delta x]$ and the tangent line. Notice that $\epsilon \to 0$ as $\Delta x \to 0$.

A function $f(x, y)$ is **differentiable** at (a, b) if and only if we can write

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

where ϵ_1 and ϵ_2 are both functions of Δx and Δy, and $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.



Lines of the same color represent a secant line on the square $[a, a + \Delta x] \times [b, b + \Delta y]$ and a tangent line in same xy direction.

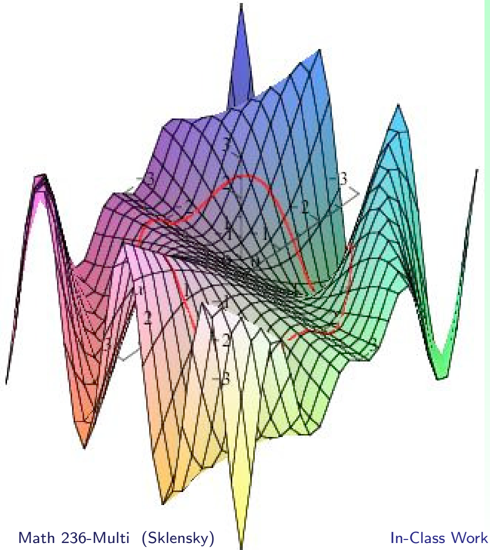
For $f(x, y)$ to be differentiable at (a, b), the slope of each secant line on this square must approach the slope of the corr. tangent line.

Fortunately, we are able to only use two ϵs, ϵ_1 and ϵ_2.

Let $f(x, y) = e^{-x^2-y^2}$.

1. Find the equation of the plane tangent to $z = f(x, y)$ at $(1, -2, e^{-5})$.
2. Use a linear approximation of $f(x, y)$ to approximate f at $(0.96, -1.97)$.

If $f(x, y) = x\cos(xy)$ models the surface of a mountain, and if $r(t) = < 2\cos(t), 2\sin(t) >$ models just the xy-coordinates of a circular path on the mountain. Then $f \circ r(t)$ gives the altitude of any point on that circular path.



Question: How is the elevation changing at $t = \frac{\pi}{2}$?

Use the chain rule to find the indicated derivatives:

(a) $g'(t)$, where

$$g(t) = f(x(t), y(t)) \qquad f(x, y) = \sqrt{x^2 + y^2}$$
$$x(t) = \sin(t) \qquad y(t) = t^2 + 2$$

(b) $\dfrac{\partial g}{\partial u}$, where

$$g(u, v) = f(x(u, v), y(u, v)) \qquad f(x, y) = 4x^2y^3$$
$$x(u, v) = u^3 - v\sin(u) \qquad y(u, v) = 4u^2$$