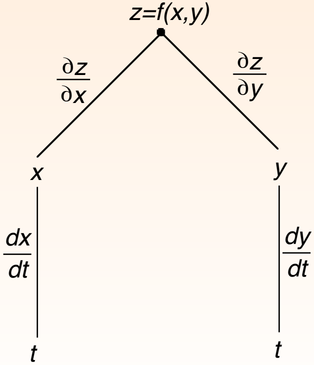
1.(a) Use the chain rule to find $g'(t)$, where

$$g(t) = f(x(t), y(t)) \qquad f(x,y) = \sqrt{x^2 + y^2}$$

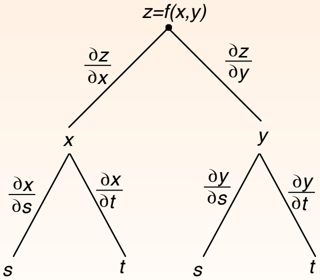
$$x(t) = \sin(t) \qquad y(t) = t^2 + 2$$



$$\begin{aligned} g'(t) &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= \left(\frac{1}{2}(x^2+y^2)^{-1/2}(2x)\right)\left(\cos(t)\right) \\ &\quad +\left(\frac{1}{2}(x^2+y^2)^{-1/2}(2y)\right)\left(2t\right) \\ &= \frac{x\cdot\cos(t)}{\sqrt{x^2+y^2}} + \frac{y\cdot 2t}{\sqrt{x^2+y^2}} \\ &= \frac{\sin(t)\cos(t) + (t^2+2)(2t)}{\sqrt{\sin^2(t) + (t^2+2)^2}} \end{aligned}$$

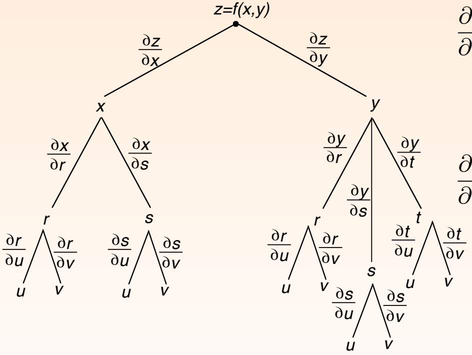
1(b) Use the chain rule to find $\dfrac{\partial g}{\partial u}$, where

$$g(u,v) = f(x(u,v), y(u,v)) \qquad f(x,y) = 4x^2y^3$$
$$x(u,v) = u^3 - v\sin(u) \qquad y(u,v) = 4u^2$$



$$\begin{aligned}
\frac{\partial g}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \\
&= (8xy^3)\big(3u^2 - v\cos(u)\big) + (12x^2y^2)(8u) \\
&= 8\big(u^3 - v\sin(u)\big)(4u^2)^3\big(3u^2 - v\cos(u)\big) \\
&\quad + 12\big(u^3 - v\sin(u)\big)^2(4u^2)^2(8u) \\
&= 512u^6\big(u^3 - v\sin(u)\big)\big(3u^2 - v\cos(u)\big) \\
&\quad + 1536u^5\big(u^3 - v\sin(u)\big) \\
&= 512u^5\big(u^3 - v\sin(u)\big) \\
&\quad \times\big(3u^3 - uv\cos(u) + 3\big)
\end{aligned}$$

2. Draw a tree diagram to figure out the chain rule for composite functions of the form $z = g(u, v) = f(x(r, s), y(r, s, t))$, where r, s, and t are all functions of u and v.



$$
\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial r}\frac{\partial r}{\partial u} + \frac{\partial x}{\partial s}\frac{\partial s}{\partial u}\right) \\
&\quad + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial r}\frac{\partial r}{\partial u} + \frac{\partial y}{\partial s}\frac{\partial s}{\partial u} + \frac{\partial y}{\partial t}\frac{\partial t}{\partial u}\right) \\
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial r}\frac{\partial r}{\partial v} + \frac{\partial x}{\partial s}\frac{\partial s}{\partial v}\right) \\
&\quad + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial r}\frac{\partial r}{\partial v} + \frac{\partial y}{\partial s}\frac{\partial s}{\partial v} + \frac{\partial y}{\partial t}\frac{\partial t}{\partial v}\right)
\end{aligned}
$$

3. Use the chain rule twice to find $g''(t)$ if $g(t) = f(x(t), y(t))$.

$$g'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \Longrightarrow g''(t) = \frac{d}{dt}\left(\frac{\partial f}{\partial x}\frac{dx}{dt}\right) + \frac{d}{dt}\left(\frac{\partial f}{\partial y}\frac{dy}{dt}\right).$$

Use the product rule on both products in the sum on the right:

$$g''(t) = \left[\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{dx}{dt} + \frac{\partial f}{\partial x}\cdot\frac{d^2x}{dt^2}\right] + \left[\frac{d}{dt}\left(\frac{\partial f}{\partial y}\right)\cdot\frac{dy}{dt} + \frac{\partial f}{\partial y}\cdot\frac{d^2y}{dt^2}\right]$$

Apply the chain rule to both $\frac{\partial f}{\partial x}(x(t), y(t))$ and $\frac{\partial f}{\partial y}(x(t), y(t))$:

$$\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y \partial x}\frac{dy}{dt} \qquad \frac{d}{dt}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y^2}\frac{dy}{dt}.$$

Putting it all together, I get:

$$\begin{aligned} g''(t) &= \frac{\partial^2 f}{\partial x^2}\left(\frac{dx}{dt}\right)^2 + \frac{\partial^2 f}{\partial y \partial x}\left(\frac{dx}{dt}\right)\left(\frac{dy}{dt}\right) + \frac{\partial f}{\partial x}\frac{d^2x}{dt^2} \\ &\quad + \frac{\partial^2 f}{\partial x \partial y}\left(\frac{dx}{dt}\right)\left(\frac{dy}{dt}\right) + \frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dt}\right)^2 + \frac{\partial f}{\partial y}\frac{d^2y}{dt^2} \end{aligned}$$