Where We're Headed

- Clear up some confusion over finding absolute extrema over a closed and bounded region.
- An application of optimization: Linear Regression
- Next up – Multivariate Integration!

Daily WeBWorK, Problem 1

1. Find the absolute maximum and the absolute minimum of the function $f(x, y) = 2x^3 + y^4$ on the region $\{(x, y) \big| x^2 + y^2 \leq 49\}$.

- **Find all critical points within the region**
  - Find the first partials, set them equal to 0, solve

$$\begin{aligned} f_x(x, y) = 6x^2 \quad &\Rightarrow \quad f_x(x, y) = 0 \text{ when } x = 0 \\ f_y(x, y) = 4y^3 \quad &\Rightarrow \quad f_y(x, y) = 0 \text{ when } y = 0 \end{aligned}$$

  - Thus the only critical point is $(0, 0)$. This point **does** lie within our region.

Daily WeBWorK, Problem 1, continued

1. Find the absolute maximum and the absolute minimum of the function $f(x,y) = 2x^3 + y^4$ on the region $\left\{(x,y)\middle|x^2+y^2 \leq 49\right\}$.

- **Find all critical points on the boundary of the region**
  - Parametrize the boundary
    - One parametrization of $x^2+y^2=7^2$: $x(t) = 7\cos(t)$, $y(t) = 7\sin(t)$
  - Use the parametrization to find the curve that is the portion of the surface lying above the boundary

$$g(t) = 2\big(7\cos(t)\big)^3 + \big(7\sin(t)\big)^4 = 686\cos^3(t) + 2401\sin^4(t).$$

  - Find where $g'(t) = 0$

$$g'(t) = 686(3)\cos^2(t)\big(-\sin(t)\big) + 2401(4)\sin^3(t)\big(\cos(t)\big)$$

Using Maple, $g'(t) = 0$ when $t = 0, t = \frac{\pi}{2}, t \approx \pm 0.4543$

Since Maple used inverse trig functions to solve, and thus may have missed a few, I check: $t = \pi$ and $t = 3\pi/2$ are also solutions.

Daily WeBWorK, Problem 1, continued

1. Find the absolute maximum and the absolute minimum of the function $f(x,y) = 2x^3 + y^4$ on the region $\left\{(x,y) \middle| x^2+y^2 \leq 49\right\}$.

- **Find all critical points on the boundary, continued**
  - Parametrize the boundary
    - One parametrization of $x^2+y^2=7^2$: $x(t) = 7\cos(t)$, $y(t) = 7\sin(t)$
  - Curve above boundary:

$$g(t) = 2\big(7\cos(t)\big)^3 + \big(7\sin(t)\big)^4 = 686\cos^3(t) + 2401\sin^4(t).$$

  - We found where $g'(t) = 0$

    $g'(t) = 0$ when $t = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}, \sim 0.4543, \sim -0.4543$
  - Convert back to x and y:

$$(7,0) \qquad (0,7) \qquad (-7,0) \qquad (0,-7) \qquad (6.29, 3.07) \qquad (6.29, -3.07)$$

Daily WeBWorK, Problem 1, continued

1. Find the absolute maximum and the absolute minimum of the function $f(x,y) = 2x^3 + y^4$ on the region $\left\{(x,y)\middle|x^2+y^2 \leq 49\right\}$.

- **Find all critical points within the region**
  - The only critical point in the interior of the region is the point $(0,0)$.
- **Find all critical points on the boundary**
  - There are six critical points on the boundary:

$$(7,0) \qquad (0,7) \qquad (-7,0) \qquad (0,-7) \qquad (6.29, 3.07) \qquad (6.29,-3.07)$$

- **Evaluate $f(x,y)$ at all the critical points in and on the boundary of the region**

$$\begin{array}{lllllllll} f(7,0) & = & 686 & f(0,7) & = & 2401 & f(6.29,3.07) & \approx & 586.55 \\ f(-7,0) & = & -686 & f(0,-7) & = & 2401 & f(6.29,-3.07) & \approx & 586.55 \\ f(0,0) & = & 0 & & & & & & \end{array}$$

- **Conclusion:** f has an abs max of 2401, at two points, and an abs min of -686, at one point. See today's Maple file for graph

Daily WeBWork, Problem 2

2. Find the maximum and minimum values of $f(x, y) = 6x + y$ on the ellipse $x^2 + 4y^2 = 1$.

- **Find all critical points within the region**
  - Find the first partials, set them equal to 0, solve

$$\begin{aligned} f_x(x, y) = 6 \quad &\Rightarrow \quad f_x(x, y) \neq 0 \\ f_y(x, y) = 1 \quad &\Rightarrow \quad f_y(x, y) \neq 0 \end{aligned}$$

Thus there are *no* critical points *inside* our region.

Daily WeBWorK, Problem 2, continued

2. Find the maximum and minimum values of $f(x,y) = 6x + y$ on the ellipse $x^2 + 4y^2 = 1$.

- **Find all critical points on the boundary of the region**
  - Parametrize the boundary
    - One parametrization of $x^2 + 4y^2 = 1$: $x(t) = \cos(t)$, $y(t) = \dfrac{\sin(t)}{2}$
  - Use the parametrization to find the curve that is the portion of the surface lying above the boundary

$$g(t) = 6\cos(t) + \frac{\sin(t)}{2}.$$

  - Find where $g'(t) = 0$

$$g'(t) = -6\sin(t) + \frac{\cos(t)}{2} \Rightarrow g'(t) = 0 \text{ when } \tan(t) = \frac{1}{12}$$

  Thus $g'(t) = 0$ when $t \approx 0.08, t \approx \pi + 0.08$

  - Convert back to x and y: Critical points on the boundary are:

$$(0.997, 0.04), (-0.997, -0.04)$$

Daily WeBWorK, Problem 2, continued

2. Find the maximum and minimum values of $f(x, y) = 6x + y$ on the ellipse $x^2 + 4y^2 = 1$.

- **Find all critical points within the region**
  - There are **no** critical points on the interior of the region
- **Find all critical points on the boundary**
  - There are two critical points on the boundary:

$$(0.997, 0.04), (-0.997, -0.04)$$

- **Evaluate $f(x, y)$ at all the critical points in and on the boundary of the region**

$$\begin{aligned} f(0.997, 0.04) &\approx 6(0.997) + 0.04 \approx 6.022 \\ f(-0.997, -0.04) &\approx 6(-0.997) - 0.04 \approx -6.022 \end{aligned}$$

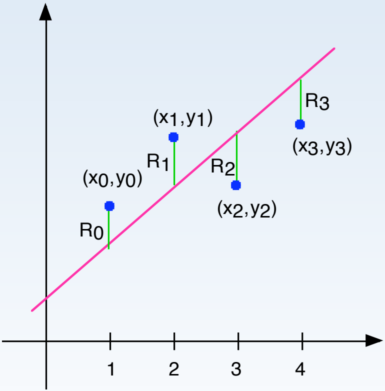
- **Conclusion:** f has an abs max of about 6 and an abs min of about -6. See today's Maple file for graph.

Where We're Headed

- Clear up some confusion over finding absolute extrema over a closed and bounded region.
- An application of optimization: Linear Regression
- Next up – Multivariate Integration!

A standard statistics problem:

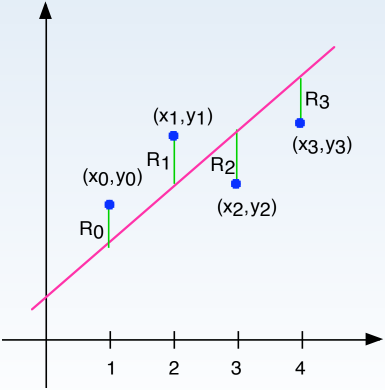
Given a bunch of data points, find a line that minimizes the total (vertical) distance between between those data points and the line.



(For each x_i, look at the distance between the y-value predicted by the line, and the actual y-value of the data.)

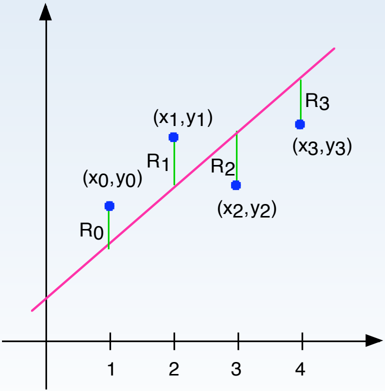
A standard statistics problem:

We want to minimize the overall distance of the points from the line, using these vertical distances, called residuals. Since we don't want negative distance to cancel out positive distance, we will instead minimize the sum of the squares of these *residuals*.



A standard statistics problem:

The line that minimizes the sum of the squares of these *residuals* is called the linear regression line.



In Class Work

The extinct beast archaeopteryx is generally regarded to have been the first bird. Here are the lengths in centimeters of the femur and humerus for five fossil specimens that preserved both bones.

Humerus (x)	41	63	70	72	84
Femur (y)	38	56	59	64	74

1. Set up the sum of the least squares that we need to minimize to find the least-squares regression line.

2. Using Maple, find the regression line.

    E:= *enter sum you found* [return]
    solve(diff(E,a)=0, diff(E,b)=0)

3. If an archaeopteryx had a humerus of length 69 cm, how long would you predict that its femur was?

Solutions

Humerus (cm) (x)	41	63	70	72	84
Femur (cm) (y)	38	56	59	64	74

1. Set up the sum that we will minimize to find the linear regression line.

   Let $x =$ humerus length (in cm), $y =$ femur length (in cm)

   Goal: find a line $y = ax + b$ that best fits the data.

Data Point	y-coord of point on Line	Residual
(41,38)	41a+b	41a+b-38
(63,56)	63a+b	63a+b-56
(70,59)	70a+b	70a+b-59
(72,64)	72a+b	72a+b-64
(84,74)	84a+b	84a+b-74

   Let $f(a, b)$ be the sum of the squares of the residuals.

$$f(a,b) = (41a+b-38)^2 + (63a+b-56)^2 + (70a+b-59)^2 + (72a+b-64)^2 + (84a+b-74)^2.$$

Solutions

For (2) and (3), see Maple solutions file!

Where We're Headed

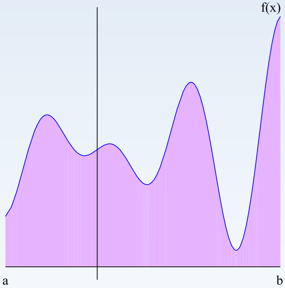
- Clear up some confusion over finding absolute extrema over a closed and bounded region.
- An application of optimization: Linear Regression
- Next up – Multivariate Integration!

Motivating Double Integrals

Question: What does $\int_a^b f(x)dx$ measure?

Motivating Double Integrals

Question: What does $\int_a^b f(x)dx$ measure?



signed area between the curve and the x-axis over the interval $[a, b]$.

Motivating Double Integrals

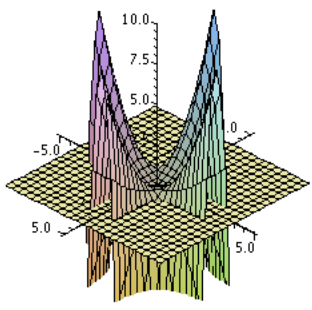
Question: We've done multivariate differentiation. What about integration?

That is, if we think of $\iint \underline{\hspace{2cm}} dA$ as being some sort of as yet undefined multivariate integral and R as a region in the xy-plane, then what might we mean by $\iint_R f(x, y)\, dA$?

Motivating Double Integrals

Question: We've done multivariate differentiation. What about integration?

That is, if we think of $\iint \_\_\_\_\_\_ dA$ as being some sort of as yet undefined multivariate integral and R as a region in the xy-plane, then what might we mean by $\iint_R f(x,y)\,dA$?



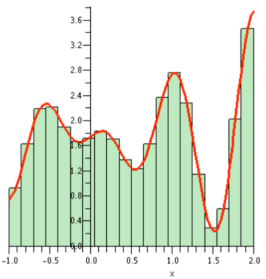
Signed volume?? between the surface and the xy-plane over the region R of the plane.

Recall from Calc 1:

To approximate the area under a positive-valued function $y = f(x)$ over the interval $I = [a, b]$, we

- **Partition the interval** $[a, b]$ into n smaller subintervals.

  Width of ith subinterval $= \Delta x_i$.



- **Pick evaluation points**, one from each subinterval. Call these c_i.

- **Add up the areas of the rectangles.**

  Base $= \Delta x_i$, height $= f(c_i) \Rightarrow$

$$A \approx \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

To find the area exactly

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

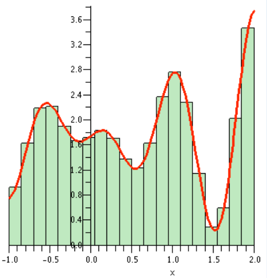
Recall from Calc 1:

To approximate the area under a positive-valued function $y = f(x)$ over the interval $I = [a, b]$, we

- **Partition the interval** $[a, b]$ into n smaller subintervals.

  Width of ith subinterval $= \Delta x_i$.

- **Pick evaluation points**, one from each subinterval. Call these c_i.



- **Add up the areas of the rectangles.**

  Base $= \Delta x_i$, height $= f(c_i) \Rightarrow$

  $$A \approx \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

To find the area exactly

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

Recall from Calc 1:

To approximate the area under a positive-valued function $y = f(x)$ over the interval $I = [a, b]$, we

- **Partition the interval** $[a, b]$ into n smaller subintervals.

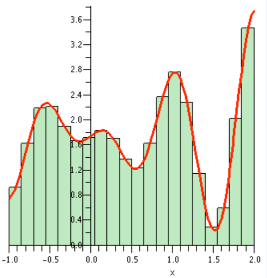
  Width of ith subinterval $= \Delta x_i$.

- **Pick evaluation points**, one from each subinterval. Call these c_i.

- **Add up the areas of the rectangles.**

  Base $= \Delta x_i$, height $= f(c_i) \Rightarrow$

  $$A \approx \sum_{i=1}^{n} f(c_i)\Delta x_i.$$



To find the area exactly

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

Recall from Calc 1:

To approximate the area under a positive-valued function $y = f(x)$ over the interval $I = [a, b]$, we

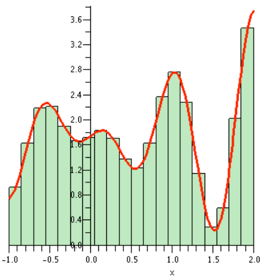
- **Partition the interval** $[a, b]$ into n smaller subintervals.

  Width of ith subinterval $= \Delta x_i$.

- **Pick evaluation points**, one from each subinterval. Call these c_i.

- **Add up the areas of the rectangles.**
  Base $= \Delta x_i$, height $= f(c_i) \Rightarrow$

$$A \approx \sum_{i=1}^{n} f(c_i)\Delta x_i.$$



To find the area exactly

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

Extend to functions of two variables, f(x, y)

We want to approximate the volume under the postiive-valued function $z = f(x, y)$ over the rectangle $R = [a, b] \times [c, d]$:

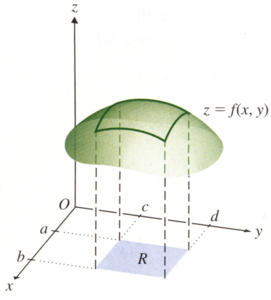


Figure 13.3

Volume under the surface $z = f(x, y)$.

Extend to $\mathbf{f(x, y)}$

- **Partition the rectangle $[\mathbf{a}, \mathbf{b}] \times [\mathbf{c}, \mathbf{d}]$ into n rectangles**, by partitioning both $[a, b]$ and $[c, d]$. Usually we'll partition both $[a, b]$ and $[c, d]$ into k subintervals, giving a total of k^2 rectangles each of area ΔA.

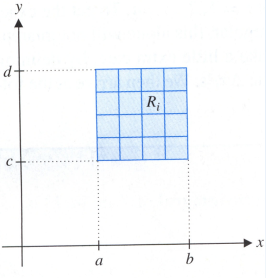


Figure 13.4a
Partition of R.

-
-
-

Extend to $\mathbf{f(x, y)}$

- **Partition the rectangle** $[\mathbf{a}, \mathbf{b}] \times [\mathbf{c}, \mathbf{d}]$ **into** n **rectangles**
- **Pick evaluation points**, one from each subinterval. Call these points (u_i, v_i).
-
-

Extend to $\mathbf{f(x, y)}$

- **Partition the rectangle $[a, b] \times [c, d]$ into n rectangles**
- **Pick evaluation points(u_i, v_i).**
- **Add up the (signed) volumes of the boxes that have as their base area ΔA and as their height $f(u_i, v_9)$.**

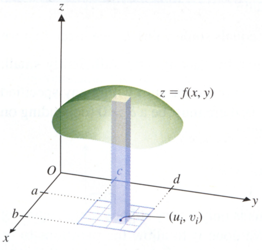


Figure 13.4b
Approximating the volume above R_i by a rectangular box.

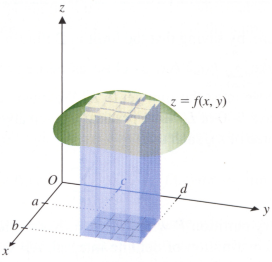


Figure 13.4c
Approximate volume.

Extend to $\mathbf{f(x, y)}$

- **Partition the rectangle $[a, b] \times [c, d]$ into n rectangles**
- **Pick evaluation points(u_i, v_i).**
- **Add up the (signed) volumes of the boxes**
- **Take the limit as the number of boxes approaches infinity (more specifically, as the diagonal of the largest base approaches zero).**

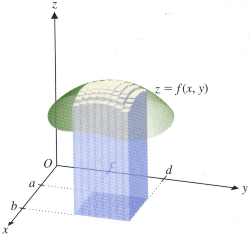


Figure 13.4d
Approximate volume.