Where We're Headed

- Last time, discussed:
  - Will use
  $$\iint_R f(x,y)\,dA$$
  as notation to represent the signed volume between $f(x,y)$ and the xy-plane over a region R.
- Today:
  - Clarifying what we mean by integration over $\mathbb{R}^2$
  - Fubini's Theorem
    - Like the FTC, this allows us to evaluate our new quantity, signed volume$=\iint_R f(x,y)\,dA$, using an old quantity, $\int f$.

Riemann Sums for Signed Volume

Goal: approximate $\iint_R f(x,y)\,dA$, the signed volume under the positive-valued function $z = f(x,y)$ over the rectangle $R = [a,b] \times [c,d]$:

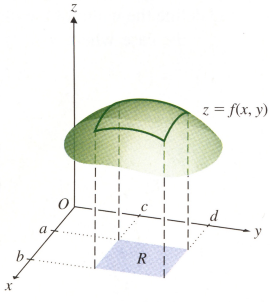

Figure 13.3
Volume under the surface $z = f(x, y)$.

Extend to $\mathbf{f(x, y)}$

- **Partition the rectangle $[\mathbf{a}, \mathbf{b}] \times [\mathbf{c}, \mathbf{d}]$ into n rectangles**, by partitioning both $[a, b]$ and $[c, d]$. Usually we'll partition both $[a, b]$ and $[c, d]$ into k subintervals, giving a total of k^2 rectangles each of area ΔA.

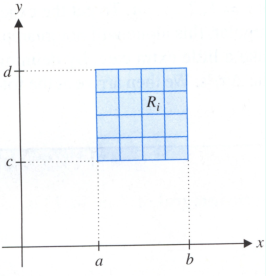

Figure 13.4a
Partition of R.

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Extend to $\mathbf{f(x, y)}$

- **Partition the rectangle $[\mathbf{a}, \mathbf{b}] \times [\mathbf{c}, \mathbf{d}]$ into n rectangles**
- **Pick evaluation points**, one from each subinterval. Call these points (u_i, v_i).
-
-

Extend to $f(x, y)$

- **Partition the rectangle $[a, b] \times [c, d]$ into n rectangles**
- **Pick evaluation points(u_i, v_i).**
- **Add up the (signed) volumes of the boxes that have as their base area ΔA and as their height $f(u_i, v_9)$.**

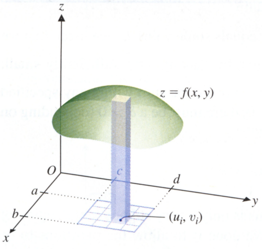


Figure 13.4b
Approximating the volume above R_i by a rectangular box.

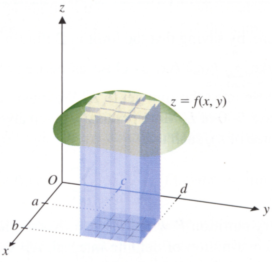


Figure 13.4c
Approximate volume.

Extend to $f(x, y)$

- **Partition the rectangle $[a, b] \times [c, d]$ into n rectangles**
- **Pick evaluation points(u_i, v_i).**
- **Add up the (signed) volumes of the boxes**
- **Take the limit as the number of boxes approaches infinity (more specifically, as the diagonal of the largest base approaches zero).**

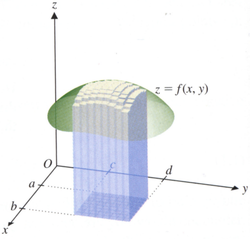
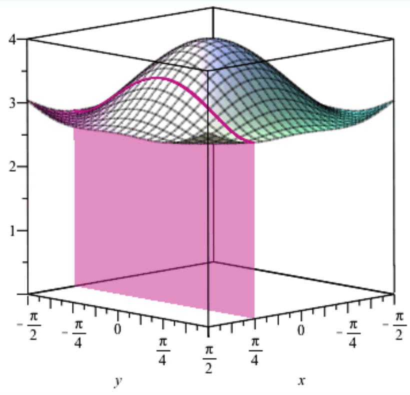


Figure 13.4d
Approximate volume.

Idea from Calc 2: Volume by Cross-Sections – Fixing x

To find $\iint_R f(x,y)\,dA =$Signed Volume over $R : [a,b] \times [c,d]$:

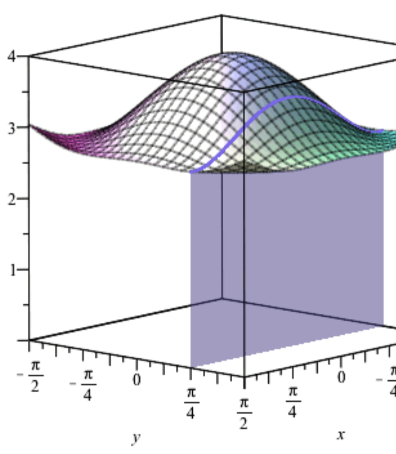
- For each x, take the cross-sectional signed area $A(x)$
- From Calc 2: $V = \int_a^b A(x)\,dx$
- But for any fixed x, $A(x) = \int_c^d f(x,y)\,dy$.
  (Keep in mind, only y acts as a variable; x is fixed)
- Thus $V = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$



Fixing y

Idea from Calc 2: Volume by Cross-Sections – Fixing x

- For each y, take the cross-sectional signed area $A(y)$
- Then $V = \int_a^b A(y)\, dy$
- For any fixed y,
  $A(y) = \int_a^b f(x, y)\, dx.$
  (Keep in mind, only x acts as a variable; y is fixed)
- Thus
  $V = \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy$



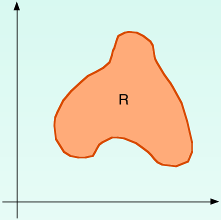
In Class Work

1. Find the volume below the surface $z = 1 + x + y$ and above the rectangle $R = \{(x, y) | 0 \leq x \leq 2, 0 \leq y \leq 3\}$ in the xy-plane.

2. Find the volume below the surface $z = y^3 e^x$ and above the rectangle $R : [-1, 1] \times [0, 2]$.

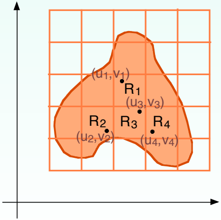
Moving beyond rectangular regions

What if our region R isn't a rectangle?



Partition the region into subrectangles.

Only consider those rectangles that lie within R.



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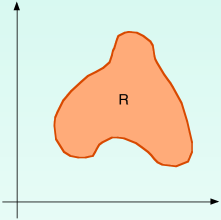

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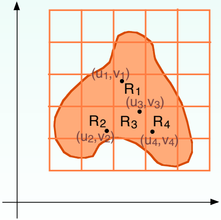

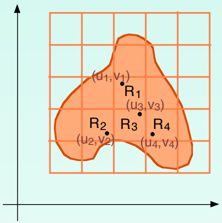

The smaller the rectangles, the more fit entirely into the region.

As we take the limit, we'll get the whole region, and the whole signed volume.

Thus we can define $\iint_R f(x, y)\, dA$ even if R isn't a rectangle.

Solutions to In Class Work

1. Find the volume below the surface $z = 1 + x + y$ and above the rectangle $R = \{(x, y)|0 \leq x \leq 2, 0 \leq y \leq 3\}$ in the xy-plane.

$$\begin{aligned}
\text{Signed Volume} &= \iint_R 1 + x + y \; dA \\
&= \int_0^2 \left(\int_0^3 1 + x + y \; dy \right) \; dx \\
&= \int_0^2 y + xy + \frac{1}{2}y^2 \Big|_0^3 \; dx \\
&= \int_0^2 (3 + 3x + \frac{9}{2}) - (0) \; dx = \int_0^2 \frac{15}{2} + 3x \; dx \\
&= \left(\frac{15}{2}x + \frac{3}{2}x^2 \right) \Big|_0^2 = (15 + 6) - (0 + 0) = 21
\end{aligned}$$

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Solutions to In Class Work

2. Find the volume below the surface $z = y^3 e^x$ and above the rectangle $R : [-1, 1] \times [0, 2]$.

$$\begin{aligned}
\text{Signed Volume} &= \iint_R y^3 e^x \, dA \\
&= \int_{-1}^{1} \left(\int_0^2 y^3 e^x \, dy \right) \, dx \\
&= \int_{-1}^{1} \frac{y^4}{4} e^x \Big|_0^2 \, dx \\
&= \int_{-1}^{1} \frac{16}{4} e^x - 0 \, dx = \int_{-1}^{1} 4e^x \, dx \\
&= 4e^x \big|_{-1}^{1} = 4e - \frac{4}{e}
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