Where We're Headed:

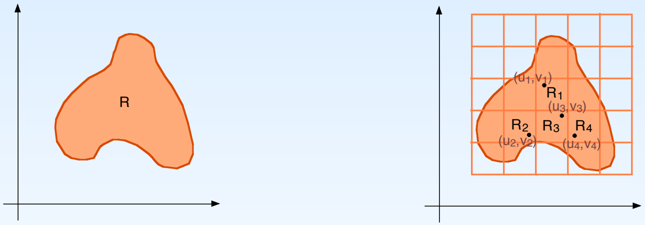
- Where we've been:
  - $\iint_R f(x,y)\,dA \overset{def}{=}$ signed volume btwn $z = f(x,y)$ and xy-plane.
  - **Fubini's Theorem:** If R is rectangular region $[a,b] \times [c,d]$:

    $$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

    - Saw an example; worked with on WeBWorK
  - Intro to ideas of integrating over non-rectangular regions
  - **Fubini's Theorem extends to non-rectangular regions!**
- Next up:
  - Continue to expand double-integration to more interestingly-shaped regions.

Recall: Over Non-Rectangular Regions

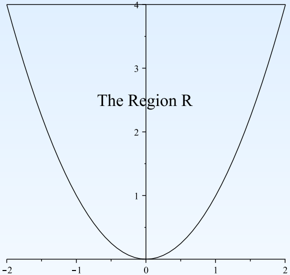
- Even if our region has a more interesting shape, partition it into rectangular sub-regions.
- Only consider those rectangles that fit entirely within our region.



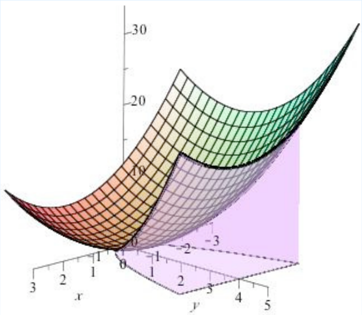
- Approximate the signed volume of our surface with boxes that sit above (or below) these rectangles.
- Take the limit as the number of rectangles approaches infinity.

Example: Double Integral over a region which is not a Rectangle

Let R be the region in the xy-plane bounded above by $y = 4$ and below by $y = x^2$. Find $\iint_R x^2 + y^2 \, dA$.



The Region R



The surface $z = x^2 + y^2$ sitting above R and the enclosed volume

Recall: Fubini's Theorem for General Regions

Even if R is a not-necessarily rectangular region in the xy-plane, there is an extension of Fubini's Theorem.

- If $a \leq x \leq b$ and $g(x) \leq y \leq h(x)$ on $[a, b]$, then it turns out that our double integral becomes an iterated integral:

$$\iint_R f(x,y)\, dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\, dy\, dx.$$

- Similarly, if $a \leq y \leq b$ and $g(y) \leq x \leq h(y)$ on $[a, b]$, then

$$\iint_R f(x,y)\, dA = \int_a^b \int_{g(y)}^{h(y)} f(x,y)\, dx\, dy.$$

In Class Work

1. Find the signed volume between the surface $z = 1 + x + y$ and the region R in the xy-plane bounded by the graphs $x = 1, y = 0, y = x^2$.

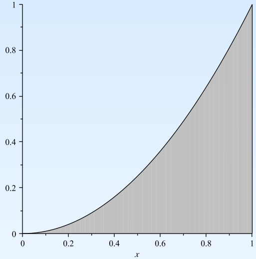
2. Find the signed volume between the surface $z = e^{-x^2}$ and the triangle R in the xy-plane bounded by the x-axis, the line $x = 1$, and the line $y = x$.

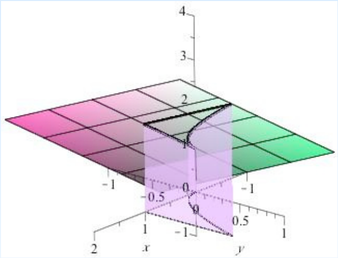
3. (a) Try to evaluate $\int_0^\pi \int_x^\pi \frac{\sin(y)}{y}\, dy\, dx$ as it's written. What happens?
   (b) Sketch the region we're integrating over.
   (c) Reverse the order of integration (using the sketch you developed in (b)), and try to evaluate the integral. Is this way more effective than the first?

4. Find the volume of portion of the solid bounded by the cylinders $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$ that lies in the first octant.

Solutions

1. Find the signed volume between the surface $z = 1 + x + y$ and the region R in the xy-plane bounded by the graphs $x = 1, y = 0, y = x^2$.





Using sketch of R, shown on the left, notice that we can either say that

$$0 \le x \le 1 \text{ and } 0 \le y \le x^2$$

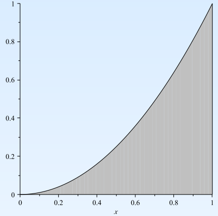
or

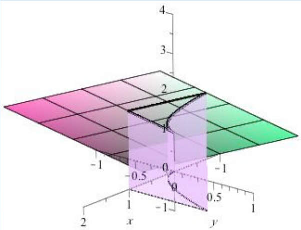
$$0 \le y \le 1 \text{ and } \sqrt{y} \le x \le 1.$$

I will choose to use $0 \le x \le 1$ and $0 \le y \le x^2$

Solutions

1. Find the signed volume between the surface $z = 1 + x + y$ and the region R in the xy-plane bounded by the graphs $x = 1, y = 0, y = x^2$.





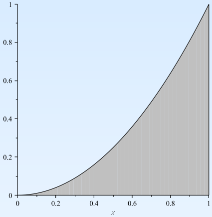
Choosing to use $0 \leq x \leq 1$ and $0 \leq y \leq x^2$

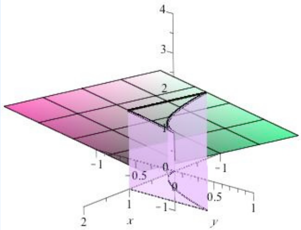
Thus by Fubini's Theorem,

$$V = \iint_R 1 + x + y \, dA = \int_0^1 \left(\int_0^{x^2} 1 + x + y \, dy \right) \, dx.$$

Solutions

1. Find the signed volume between the surface $z = 1 + x + y$ and the region R in the xy-plane bounded by the graphs $x = 1, y = 0, y = x^2$.



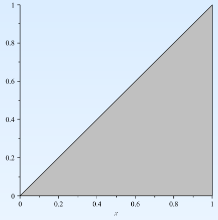


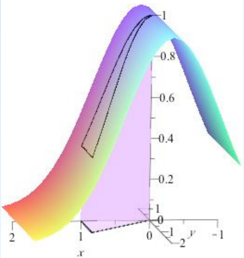
Thus by Fubini's Theorem,

$$
\begin{aligned}
V &= \iint_R 1 + x + y \, dA = \int_0^1 \left(\int_0^{x^2} 1 + x + y \, dy \right) dx \\
&= \int_0^1 \left(y + xy + \frac{y^2}{2} \Big|_0^{x^2} \right) dx \\
&= \int_0^1 x^2 + x^3 + \frac{1}{2} x^4 \, dx = \ldots = \frac{41}{60}
\end{aligned}
$$

Solutions

2. Find the volume below the surface $z = e^{-x^2}$ and above the triangle R in the xy-plane bounded by the x-axis, the line $x = 1$, and the line $y = x$.





Again, using sketch of region R on left, write the region in two ways :

$$0 \leq x \leq 1 \text{ and } 0 \leq y \leq x$$

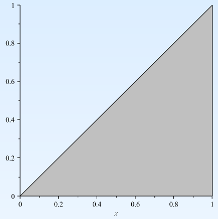
or

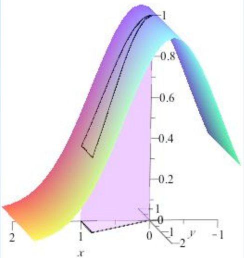
$$0 \leq y \leq 1 \text{ and } y \leq x \leq 1.$$

I will choose to use $0 \leq x \leq 1$ and $0 \leq y \leq x$

Solutions

2. Find the volume below the surface $z = e^{-x^2}$ and above the triangle R in the xy-plane bounded by the x-axis, the line $x = 1$, and the line $y = x$.





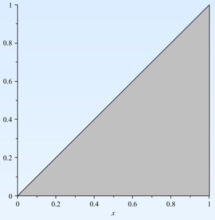
Choosing to use $0 \le x \le 1$ and $0 \le y \le x$

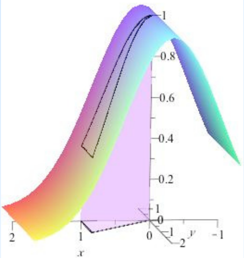
Thus by Fubini's Theorem,

$$V = \iint_R e^{-x^2}\, dA = \int_0^1 \left(\int_0^x e^{-x^2}\, dy \right)\, dx$$

Solutions

2. Find the volume below the surface $z = e^{-x^2}$ and above the triangle R in the xy-plane bounded by the x-axis, the line $x = 1$, and the line $y = x$.





Thus by Fubini's Theorem,

$$\begin{aligned} V &= \iint_R e^{-x^2}\, dA = \int_0^1 \left(\int_0^x e^{-x^2}\, dy \right)\, dx \\ &= \int_0^1 \left(ye^{-x^2} \Big|_0^x \right)\, dx = \int_0^1 xe^{-x^2}\, dx \\ &\overset{u-sub}{=} \ldots = -\frac{1}{2}\left(\frac{1}{e} - 1\right) = -\frac{1}{2e} + \frac{1}{2} \end{aligned}$$

Solutions

3.(a)

Try to evaluate $\int_0^\pi \int_x^\pi \frac{\sin(y)}{y}\, dy\, dx$ as it's written. What happens?

$$\int_0^\pi \int_x^\pi \frac{\sin(y)}{y}\, dy\, dx = \int_0^\pi \int_x^\pi \frac{1}{y} \sin(y)\, dy\, dx$$

I can already see that neither substitution nor integration by parts is going to work wonders on this integral.

When I try to do the inner integral on Maple, I get some function called "Si(y)", which I've never heard of ... or at least, it rings no bells.

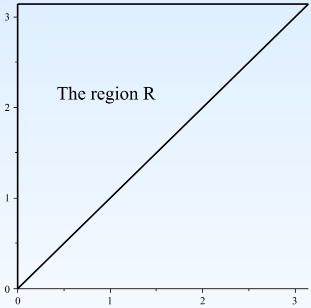
Solutions

3(b) Sketch the region we're integrating over.

In order to sketch the region, I need to look at what intervals I'm integrating over.

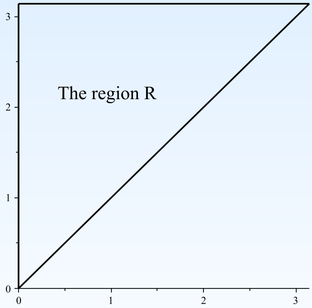
Looking at the integral $\int_0^\pi \left(\int_x^\pi \frac{\sin(y)}{y}\, dy \right) \quad dx$, we see that $0 \leq x \leq \pi$ and $x \leq y \leq \pi$.

In other words, as x goes from 0 to π, y goes from the diagonal lines $y = x$ up to the horizontal line $y = \pi$.



Solutions

3(c) Reverse the order of integration (using the sketch you developed in (b)), and try to evaluate the integral. Is this way more effective than the first?



Looking at the region R, I can see that we can also say that

$$0 \leq y \leq \pi \text{ and } 0 \leq x \leq y.$$

We can thus rewrite the integral:

$$\int_0^{\pi} \left(\int_0^{y} \frac{\sin(y)}{y}\, dx \right) dy.$$

Because this time we're integrating with respect to x first, and because our integrand is constant with respect to x, this is suddenly much easier!

Solutions

3(c) Reverse the order of integration (using the sketch you developed in (b)), and try to evaluate the integral. Is this way more effective than the first?

We have just found that

$$\int_0^\pi \int_0^y \frac{\sin(y)}{y}\, dy\, dx = \int_0^\pi \left(\int_0^y \frac{\sin(y)}{y}\, dx \right)\, dy.$$

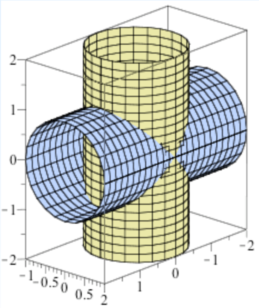
The integral on the left is a very difficult integral for us. As for the integral on the *right* ...

$$\begin{aligned}
\int_0^\pi \left(\int_0^y \frac{\sin(y)}{y}\, dx \right)\, dy &= \int_0^\pi \left(\frac{\sin(y)}{y} \int_0^y 1\, dx \right)\, dy \\
&= \int_0^\pi \left(\frac{\sin(y)}{y} (x) \Big|_0^y \right)\, dy \\
&= \int_0^\pi \frac{\sin(y)}{y} (y-0)\, dy = \int_0^\pi \sin(y)\, dy \\
&= -\cos(y)\Big|_0^\pi = -(-1) - (-1) = 2
\end{aligned}$$

Solutions

4. Find the volume of portion of the solid bounded by the cylinders $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$ that lies in the first octant.

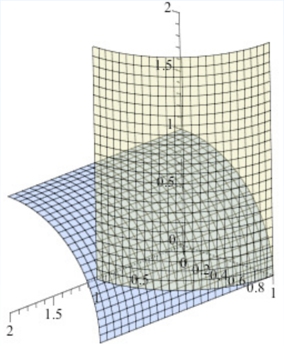
$x^2 + y^2 = 1$ is a cylinder of radius 1 extending upwards along the z-axis (in pale yellow), while $y^2 + z^2 = 1$ is a cylinder of radius 1 extending along the x-axis (in skyblue).



Solutions

4. Find the volume of portion of the solid bounded by the cylinders $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$ that lies in the first octant.

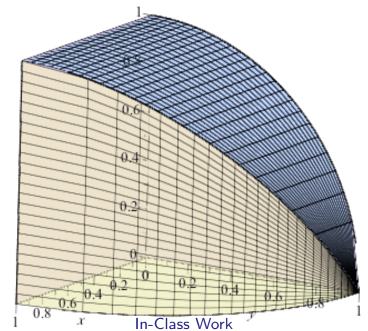
We only want to consider the portion that lies in the first octant; that is, where $x \geq 0$, $y \geq 0$, and $z \geq 0$:



Solutions

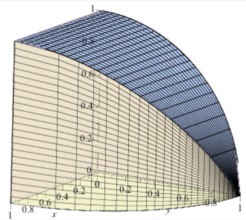
4. Find the volume of portion of the solid bounded by the cylinders $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$ that lies in the first octant.

We want the portion that's enclosed by the *xz*-plane and the *yz*-plane as two sides, the portion of the top-half of the cylinder $y^2 + z^2 = 1$ that lies above the quarter of the unit circle that lies in the first quadrant of the *xy*-plane, and that has as its third "side" the portion of the vertical cylinder that goes from the *xy*-plane to this top portion.



Solutions

4. Find the volume of portion of the solid bounded by the cylinders $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$ that lies in the first octant.



Our solid lies under the top half of $y^2 + z^2 = 1$, over the region that is the quarter unit-circle that lies in the first quadrant of the xy-plane.

Thus the function we're integrating is

$$z = \sqrt{1 - y^2},$$

and the region we're integrating over is

$$R : 0 \leq x \leq 1, 0 \leq y \leq \sqrt{1 - x^2} \;\; or \;\; R : 0 \leq y \leq 1, 0 \leq x \leq \sqrt{1 - y^2}.$$

Solutions

4. Find the volume of portion of the solid bounded by the cylinders $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$ that lies in the first octant.

The function we're integrating is

$$z = \sqrt{1 - y^2},$$

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$$R : 0 \leq x \leq 1, 0 \leq y \leq \sqrt{1 - x^2} \quad or \quad R : 0 \leq y \leq 1, 0 \leq x \leq \sqrt{1 - y^2}.$$

Thus the volume is given by

$$V = \int_0^1 \left(\int_0^{\sqrt{1-x^2}} \sqrt{1 - y^2}\, dy \right) dx$$

or

$$V = \int_0^1 \left(\int_0^{\sqrt{1-y^2}} \sqrt{1 - y^2}\, dx \right) dy.$$

Solutions

4. Find the volume of portion of the solid bounded by the cylinders $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$ that lies in the first octant.

$$V = \int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\, dy\, dx \text{ or } V = \int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{1-y^2}\, dx\, dy.$$

Which would we rather do?

Solutions

4. Find the volume of portion of the solid bounded by the cylinders $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$ that lies in the first octant.

$$
\begin{aligned}
V &= \int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{1-y^2}\,dx\,dy \\
&= \int_0^1 x\sqrt{1-y^2}\Big|_0^{\sqrt{1-y^2}}\,dy \\
&= \int_0^1 1 - y^2\,dy \\
&= y - \frac{1}{3}y^3\Big|_0^1 \\
&= 1 - \frac{1}{3} = \frac{2}{3}
\end{aligned}
$$