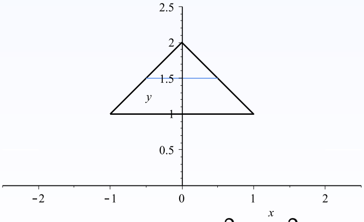
WeBWorK #2

Write down the iterated integral which expresses the surface area of $z = y^7 \cos^2(x)$ over the triangle with vertices $(-1,1),(1,1),(0,2)$:

$$\int_a^b \int_{f(y)}^{g(y)} \sqrt{h(x,y)}\, dx\, dy$$

First, sketch the region:



- $y \in [1,2]$.
- x goes from the left line to the right line.
- Left: $y = x + 2$, or $x = y - 2$.
- Right: $y = -x + 2$, or $x = 2 - y$.
- Integrand: $\sqrt{f_x^2 + f_y^2 + 1}$

$$f_x^2 + f_y^2 + 1 = [2y^7 \cos(x) \sin(x)]^2 + [7y^6 (\cos(x))^2]^2 + 1$$

Where we're going:

- In Chapter 14, we combine the idea of vector-valued functions ($\overrightarrow{\mathbf{f}} : \mathbb{R} \to \mathbb{R}^n$), with the idea of multivariate functions ($\overrightarrow{\mathbf{f}} : \mathbb{R}^n \to \mathbb{R}$).
- We will now look at functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $f : \mathbb{R}^3 \to \mathbb{R}^3$.

Definition:

- A **vector field** in the plane is a function $\overrightarrow{\mathbf{F}} : \mathbb{R}^2 \to V^2$ (where V^2 is the set of two-dimensional vectors).

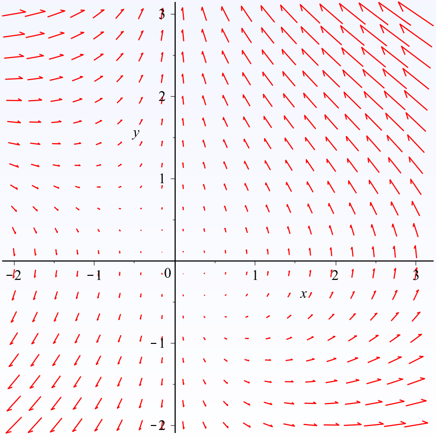
  We write
  $$\overrightarrow{\mathbf{F}}(x, y) = \langle f_1(x, y), f_2(x, y) \rangle$$
  for scalar functions $f_1,\ f_2 : \mathbb{R}^2 \to \mathbb{R}$.

- A **vector field** in space is a function $\overrightarrow{\mathbf{F}} : \mathbb{R}^3 \to V^3$.

  We write
  $$\overrightarrow{\mathbf{F}}(x, y, z) = \langle f_1(x, y, z), f_2(x, y, z), f_3(x, y, z) \rangle$$
  for scalar functions $f_1,\ f_2,\ f_3 : \mathbb{R}^3 \to \mathbb{R}$.

Question:

Suppose the vector field below shows the flow of a small portion of a river, and suppose a leaf were to fall into the water.



1. Describe the movement of the leaf, if it lands at the point $(-2, 1.25)$.
2. How does your description change if instead the leaf lands a short way away at the point $(-2, 1)$?

In Class Work

1. Find a potential function $f(x, y)$ for the vector field

   $$\overrightarrow{\mathbf{F}}(x, y) = < \cos(y) - 2 + y^2 e^{xy^2}, \cos(y) - x\sin(y) + 3y^2 + 2xye^{xy^2} > .$$

2. Find a potential function $f(x, y)$ for the vector field

   $$\overrightarrow{\mathbf{F}}(x, y) = < 2y + 2xy^2, 2x + 2y >$$

   ... if you can

Solutions

1.Find a potential function $f(x,y)$ for the vector field $\overrightarrow{\mathbf{F}}(x,y) = < \cos y - 2 + y^2 e^{xy^2}, \cos y - x \sin y + 3y^2 + 2xye^{xy^2} >$.

$$f_x = \cos(y) - 2 + y^2 e^{xy^2} \quad \Rightarrow \quad f(x,y) = \int \cos(y) - 2 + y^2 e^{xy^2}\, dx$$

$$\Rightarrow \quad f(x,y) = x\cos(y) - 2x + e^{xy^2} + g(y)$$

Solutions

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$$\Rightarrow \quad f_y = -x\sin(y) + 2yxe^{xy^2} + g'(y)$$

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$$\Rightarrow \quad f_y = -x\sin(y) + 2yxe^{xy^2} + g'(y)$$

Since we already know what f_y is, we can solve for $g(y)$:

$$f_y = \cos(y) - x\sin(y) + 3y^2 + 2xye^{xy^2} \quad \Rightarrow \quad g'(y) = \cos(y) + 3y^2$$

Solutions

1.Find a potential function $f(x, y)$ for the vector field $\overrightarrow{\mathbf{F}}(x, y) =< \cos y - 2 + y^2 e^{xy^2}, \cos y - x \sin y + 3y^2 + 2xye^{xy^2} >$.

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$$f_y = \cos(y) - x\sin(y) + 3y^2 + 2xye^{xy^2} \quad \Rightarrow \quad g'(y) = \cos(y) + 3y^2$$

$$\Rightarrow \quad g(y) = \sin(y) + y^3 + c$$

Thus *one* potential function for $\overrightarrow{\mathbf{F}}$ is

$$f(x, y) = x\cos(y) - 2x + e^{xy^2} + \sin(y) + y^3.$$

Solutions

2.Find a potential function $f(x, y)$ for the vector field $\overrightarrow{\mathbf{F}}(x, y) =< 2y + 2xy^2, 2x + 2y >$ ... if you can.

$$f_x = 2y + 2xy^2 \Rightarrow f(x, y) = \int 2y + 2xy^2 \, dx$$

$$\Rightarrow f(x, y) = 2xy + x^2y^2 + g(y)$$

Thus there is no potential function for this particular vector field.

Note: Not every vector field has an associated potential function.

Solutions

2.Find a potential function $f(x, y)$ for the vector field $\overrightarrow{\mathbf{F}}(x, y) =< 2y + 2xy^2, 2x + 2y >$ ... if you can.

$$\begin{aligned} f_x = 2y + 2xy^2 \quad &\Rightarrow \quad f(x, y) = \int 2y + 2xy^2 \; dx \\ &\Rightarrow \quad f(x, y) = 2xy + x^2y^2 + g(y) \\ &\Rightarrow \quad f_y = 2x + 2x^2y + g'(y) \end{aligned}$$

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$$\begin{aligned} f_y = 2x + 2y \quad &\Rightarrow \quad 2x + 2y = 2x + 2x^2y + g'(y) \\ &\Rightarrow \quad g'(y) = 2y - 2x^2y \end{aligned}$$

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$$\Rightarrow \quad f(x, y) = 2xy + x^2y^2 + g(y)$$

$$\Rightarrow \quad f_y = 2x + 2x^2y + g'(y)$$

$$f_y = 2x + 2y \quad \Rightarrow \quad 2x + 2y = 2x + 2x^2y + g'(y)$$

$$\Rightarrow \quad g'(y) = 2y - 2x^2y$$

but this is impossible, because $g'(y)$ must be a function of only y.

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