WeBWorK Problem 2

2. Evaluate the line integral $\int_{\mathcal{C}} x^3 z \, ds$ where $\mathcal{C}$ is the line segment from $(0, 7, 4)$ to $(4, 3, 2)$.

If we parametrize $\mathcal{C}$ with $\overrightarrow{\mathbf{r}}(t) = \langle x(t), y(t), z(t) \rangle$ on $a \leq t \leq b$, then

$$\int_{\mathcal{C}} x^3 z \, ds = \int_a^b f\big(x(t), y(t), z(t)\big) \sqrt{\big[x'(t)\big]^2 + \big[y'(t)\big]^2 + \big[z'(t)\big]^2} \, dt.$$

Parametrize $\mathcal{C}$ with
$\overrightarrow{\mathbf{r}}(t) = \langle 0 + (4-0)t, 7 + (3-7)t, 4 + (2-4)t \rangle = < 4t, 7-4t, 4-2t >$
on $0 \leq t \leq 1$.

$$\int_{\mathcal{C}} x^3 z \, ds = \int_0^1 \left[(4t)^3\right]\left[4 - 2t\right] \sqrt{4^2 + 4^2 + 2^2} \, dt = 6 \int_0^1 256t^3 - 128t^4 \, dt$$

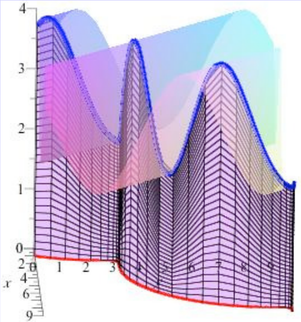
Recall: Line Integrals With Respect to Arclength

The area of the vertical surface that we create if we go straight up from the curve $\mathcal{C}$ to the surface is given by

If $\mathcal{C}$ is parametrized by $\overrightarrow{\mathbf{r}}(t)$ for $t \in [a, b]$, then

$$\int_{\mathcal{C}} f \; ds \overset{def}{=} \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i)\, \Delta s_i$$

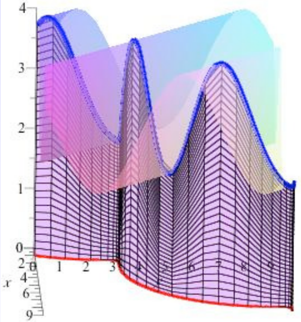
$$= \int_a^b f\left(\overrightarrow{\mathbf{r}}(t)\right) \|\overrightarrow{\mathbf{r}}\,'(t)\| \; dt$$



Other Applications of Line Integrals With Respect to Arclength

The area of the vertical surface that we create if we go straight up from the curve $\mathcal{C}$ to the surface is given by

- Signed area of the vertical surface created by connecting the curve $\mathcal{C}$ lying in the xy-plane to the portion of $z = f(x, y)$ lying directly above $\mathcal{C}$.
- Mass and Center of mass of an object in the shape of a curve – a spring or a wire, for instance
- Arclength



Work

- The work done by a constant force F to move an object a distance of d in a straight line (in the same direction as the force) is
$$W = Fd.$$
- Saw in §10.3: The work done by a constant force $\overrightarrow{\mathbf{F}}$ to move an object along the vector $\overrightarrow{\mathbf{d}}$ when $\overrightarrow{\mathbf{F}}$ doesn't necessarily point in the same direction as $\overrightarrow{\mathbf{d}}$ is
$$W = \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{d}}.$$
- Recall from Calc 1 or 2: The work done by a variable force $F(x)$ on an interval $[a, b]$ (in the same direction as the force) is
$$W = \int_a^b F(x)\, dx.$$

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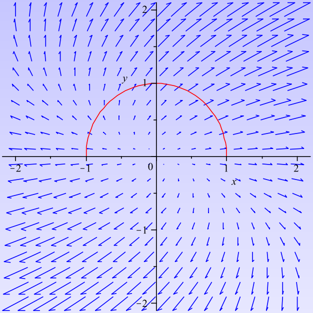
$$W = \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{d}}.$$
- Recall from Calc 1 or 2: The work done by a variable force $F(x)$ on an interval $[a, b]$ (in the same direction as the force) is
$$W = \int_a^b F(x)\, dx.$$
- What is the work done by a variable force $\overrightarrow{\mathbf{F}}(x, y)$ on a curve $\mathcal{C}$?

Work

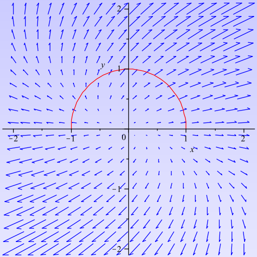
A particle moves through the force field $\overrightarrow{\mathbf{F}}$ along the **oriented** curve $\mathcal{C}$.



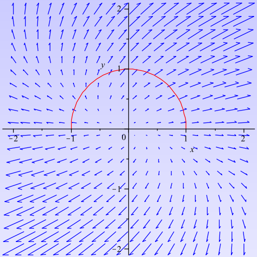
Question: How do we measure the work done *by the force field* in moving this particle along that curve? (In other words, how do we measure the contribution of the force field in getting the particle to where it ends up?)

Work

- Parametrize $\mathcal{C}$ with a v.v.f. $\overrightarrow{\mathbf{r}}(t)$.
  Be sure $\overrightarrow{\mathbf{r}}(t)$ describes $\mathcal{C}$'s orientation.
- Subdivide $\mathcal{C}$ into n subintervals.
- Each subinterval $\mathcal{C}_i$ has length Δs_i.



Work



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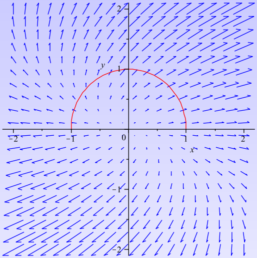
  Be sure $\overrightarrow{\mathbf{r}}(t)$ describes $\mathcal{C}$'s orientation.
- Subdivide $\mathcal{C}$ into n subintervals.
- Each subinterval $\mathcal{C}_i$ has length Δs_i.
- Pick pt $P_i(x_i, y_i, z_i) = \overrightarrow{\mathbf{r}}(t_i)$ on $\mathcal{C}_i$.
- Approx $\mathcal{C}_i$ by a vector $\overrightarrow{\mathbf{d}}_i$ tangent to $\mathcal{C}_i$ at P_i of length Δs_i.
- $\overrightarrow{\mathbf{d}}_i = \Delta s_i \left(\dfrac{\overrightarrow{\mathbf{r}}\,'(t_i)}{\|\overrightarrow{\mathbf{r}}\,'(t_i)\|} \right)$

Work



- Parametrize $\mathcal{C}$ with a v.v.f. $\vec{\mathbf{r}}(t)$.
  Be sure $\vec{\mathbf{r}}(t)$ describes $\mathcal{C}$'s orientation.
- Subdivide $\mathcal{C}$ into n subintervals.
- Each subinterval $\mathcal{C}_i$ has length Δs_i.
- Pick pt $P_i(x_i, y_i, z_i) = \vec{\mathbf{r}}(t_i)$ on $\mathcal{C}_i$.
- Approx $\mathcal{C}_i$ by a vector $\vec{\mathbf{d}}_i$ tangent to $\mathcal{C}_i$ at P_i of length Δs_i.
- $\vec{\mathbf{d}}_i = \Delta s_i \left(\dfrac{\vec{\mathbf{r}}\,'(t_i)}{\|\vec{\mathbf{r}}\,'(t_i)\|} \right)$
- Work over $\mathcal{C}_i = W_i \approx \vec{\mathbf{F}}(\vec{\mathbf{r}}(t_i)) \cdot \vec{\mathbf{d}}_i$
- $W \approx \displaystyle\sum_{i=1}^{n} \vec{\mathbf{F}}(\vec{\mathbf{r}}(t_i)) \cdot \left(\dfrac{\vec{\mathbf{r}}\,'(t_i)}{\|\vec{\mathbf{r}}\,'(t_i)\|} \right) \Delta s_i$
- $W = \displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \dfrac{\vec{\mathbf{F}}(\vec{\mathbf{r}}(t_i)) \cdot \vec{\mathbf{r}}\,'(t_i)}{\|\vec{\mathbf{r}}\,'(t_i)\|} \Delta s_i$

Work, continued

What we have so far:
The work done by a force field $\overrightarrow{\mathbf{F}}$ acting on an object moving along a curve $\mathcal{C}$ is

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\overrightarrow{\mathbf{F}}\left(\overrightarrow{\mathbf{r}}(t_i)\right) \cdot \overrightarrow{\mathbf{r}}\,'(t_i)}{\|\overrightarrow{\mathbf{r}}\,'(t_i)\|} \Delta s_i$$

Recall: If $\mathcal{C}$ is parametrized by $\overrightarrow{\mathbf{r}}(t)$ for $t \in [a, b]$ and $f : \mathbb{R}^n \to \mathbb{R}$, then

$$\begin{aligned} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\, \Delta s_i &\overset{def}{=} \int_{\mathcal{C}} f\, ds \\ &= \int_a^b f\left(\overrightarrow{\mathbf{r}}(t)\right) \|\overrightarrow{\mathbf{r}}\,'(t)\|\, dt \end{aligned}$$

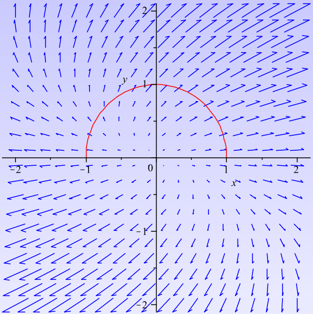
Notation:

If $\overrightarrow{\mathbf{F}}(x,y) =< F_1(x,y), F_2(x,y) >$ and $\overrightarrow{\mathbf{r}}(t) =< x(t), y(t) >$, then

$$\begin{aligned}\int_{\mathcal{C}} \overrightarrow{\mathbf{F}}(x,y)\cdot d\overrightarrow{\mathbf{r}} &= \int_{\mathcal{C}} < F_1(x,y), F_2(x,y) > \cdot < x'(t), y'(t) > \; dt \\ &= \int_{\mathcal{C}} F_1(x(t),y(t)) \boxed{x'(t)dt} + F_2(x(t),y(t)) \boxed{y'(t)\; dt} \\ &= \int_{\mathcal{C}} F_1(x,y)\; dx + F_2(x,y)\; dy\end{aligned}$$

We sometimes call these **component-wise line integrals**.

Work - Example



Find the work done by the force field

$$\vec{\mathbf{F}}(x,y) = < x+y, y >$$

acting on a object moving along the curve $\mathcal{C}$ that corresponds to the upper unit semicircle oriented counterclockwise.

Found that $W = -\dfrac{\pi}{2}$

Does it make sense that the work is negative?

Two Types of Line Integrals

- **Line Integral With Respect to Arclength**
  Let f be a continuous function defined on a smooth curve $\mathcal{C}$ given by $\overrightarrow{\mathbf{r}}(t)$ for $a \leq t \leq b$. Then the line integral of f along $\mathcal{C}$ is

$$\int_{\mathcal{C}} f\, ds = \int_a^b f\left(\overrightarrow{\mathbf{r}}(t)\right) \left\|\overrightarrow{\mathbf{r}}\,'(t)\right\| dt$$

- **Line Integral of a Vector Field along a Curve**
  Let $\overrightarrow{\mathbf{F}}$ be a continuous vector field defined on a smooth curve $\mathcal{C}$ given by $\overrightarrow{\mathbf{r}}(t)$ for $a \leq t \leq b$. Then the line integral of $\overrightarrow{\mathbf{F}}$ along $\mathcal{C}$ is

$$\int_{\mathcal{C}} \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{r}} = \int_a^b \overrightarrow{\mathbf{F}}\left(\overrightarrow{\mathbf{r}}(t)\right) \cdot \overrightarrow{\mathbf{r}}\,'(t)\, dt$$

In Class Work

Let:

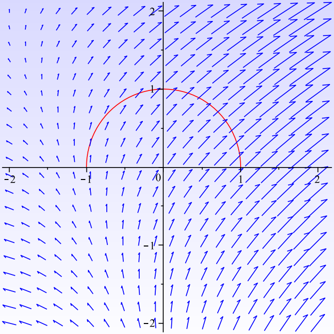
$$\overrightarrow{\mathbf{F}}(x,y) = < 2x + y + 2, x + 3 >$$

$\mathcal{C}_1$ *be the upper unit semicircle oriented counterclockwise*

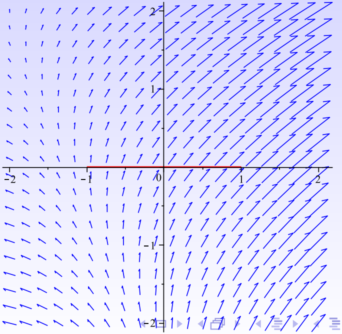
$\mathcal{C}_2$ *be the line segment from* $(1,0)$ *to* $(-1,0)$

Compute the amount of work done by $\overrightarrow{\mathbf{F}}(x,y)$ acting on an object

1. moving along $\mathcal{C}_1$



2. moving along $\mathcal{C}_2$

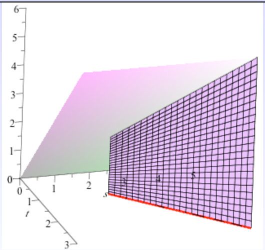
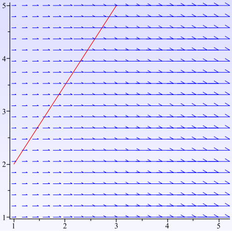


Notation:

If $\overrightarrow{\mathbf{F}}(x,y) =< F_1(x,y), F_2(x,y) >$ and $\overrightarrow{\mathbf{r}}(t) =< x(t), y(t) >$, then

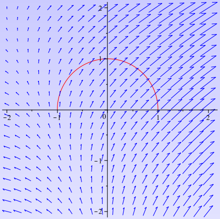
$$\begin{aligned}\int_{\mathcal{C}} \overrightarrow{\mathbf{F}}(x,y) \cdot d\overrightarrow{\mathbf{r}} &= \int_{\mathcal{C}} < F_1(x,y), F_2(x,y) > \cdot < x'(t), y'(t) > \ dt \\ &= \int_{\mathcal{C}} F_1(x(t), y(t)) x'(t) dt + F_2(x(t), y(t)) y'(t) \ dt \\ &= \int_{\mathcal{C}} F_1(x,y) \ dx + F_2(x,y) \ dy\end{aligned}$$

Let $\mathcal{C}$ be the line segment from $(1,2)$ to $(3,5)$. Use the parametrization $\overrightarrow{\mathbf{r}}(t) = < 1+2t, 2+3t >$.

From Monday	From Today
$\int_{\mathcal{C}} 2x\ ds$	$\int_{\mathcal{C}} 2x\ dx$
$\int_0^1 2x(t)\sqrt{x'(t)^2+y'(t)^2}\ dt$	$\int_{\mathcal{C}} < 2x, 0 > \cdot < x'(t), y'(t) >\ dt$
	

Solutions

1. Let $\overrightarrow{\mathbf{F}}(x,y) = < 2x + y + 2, x + 3 >$ and let $\mathcal{C}_1$ be the upper unit semicircle oriented counterclockwise. Compute the amount of work done by $\overrightarrow{\mathbf{F}}(x,y)$ acting on an object moving along $\mathcal{C}_1$.



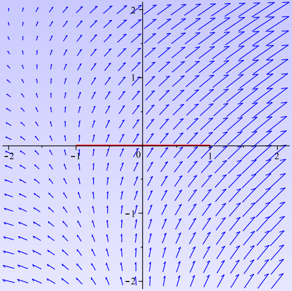
- Parametrize $\mathcal{C}_1$: $\overrightarrow{\mathbf{r}}(t) = < \cos(t), \sin(t) >$, $0 \le t \le \pi$.
- Work $= \int_{\mathcal{C}_1} \overrightarrow{\mathbf{F}}(x,y) \cdot d\overrightarrow{\mathbf{r}}$

  $\Rightarrow W = \int_0^\pi \overrightarrow{\mathbf{F}}(\cos(t), \sin(t)) \cdot \overrightarrow{\mathbf{r}}\,'(t)\,dt.$

$$
\begin{aligned}
W &= \int_0^\pi < 2\cos(t) + \sin(t) + 2, \cos(t) + 3 > \cdot < -\sin(t), \cos(t) > \; dt \\
&= \int_0^\pi -2\cos(t)\sin(t) - \sin^2(t) - 2\sin(t) + \cos^2(t) + 3\cos(t)\; dt \\
&= \int_0^\pi -\sin(2t) + \cos(2t) - 2\sin(t) + 3\cos(t)\; dt = \ldots = -4
\end{aligned}
$$

Solutions

2. Let $\overrightarrow{\mathbf{F}}(x,y) = <2x+y+2, x+3>$ and let $\mathcal{C}_2$ be the line segment from $(1,0)$ to $(-1,0)$. Compute the amount of work done by $\overrightarrow{\mathbf{F}}(x,y)$ acting on an object moving along $\mathcal{C}_2$.



- Parametrize $\mathcal{C}_2$:
  $\overrightarrow{\mathbf{r}}(t) = <1-2t, 0>, 0 \le t \le 1$
- Work $= \int_{\mathcal{C}_2} \overrightarrow{\mathbf{F}}(x,y) \cdot d\overrightarrow{\mathbf{r}}$

  $W = \int_0^1 \overrightarrow{\mathbf{F}}(1-2t, 0) \cdot \overrightarrow{\mathbf{r}}\,'(t)\, dt.$

$$\begin{aligned} W &= \int_0^1 < 2(1-2t)+0+2, 1-2t+3 > \cdot < -2, 0 > \; dt \\ &= \int_0^1 < 4-4t, 4-2t > \cdot < -2, 0 > \; dt \\ &= \int_0^1 -8+8t\; dt = -8t+4t^2\Big|^1 = -4 \end{aligned}$$