Where we've been, where we're headed:

- **Recall:**
  - On a smooth function $f(x, y)$, extrema and saddle points can only occur at a point $\big(a, b, f(a, b)\big)$ if $f_x(a, b) = 0 = f_y(a, b)$.

    Not all critical points **will** be local max, local min, or saddle point.
  - Use the 2^{nd} Derivatives Test to classify each critical point as Local Minimum, Local Maximum, Saddle Point, or none of the above (where possible).
- **We will see:**
  - Just as with functions of a single variable, we have an Extreme Value Theorem that guarantees that, under certain conditions, on a closed and bounded region f will have both an absolute maximum and an absolute minimum value.
  - Just as with functions of a single variable, the absolute max and absolute min may occur at a local max or local min, or may occur on the boundary of the closed region you're considering.

Where we're headed:

- We will look at how working with more than one variable affects how we approach finding absolute maxima and absolute minima.

- Applications of optimization are easy to think of.
- We will talk about a few less obvious ones:
  - The Method of Steepest Ascent (discuss only)
  - Linear Regression (more detail)

Recall: In Class Work from Last Time

2. Let $f(x, y) = 4xy - x^3 - 2y^2$. Find and classify all critical points of f.

   Last time found, using the Second Derivatives Test:

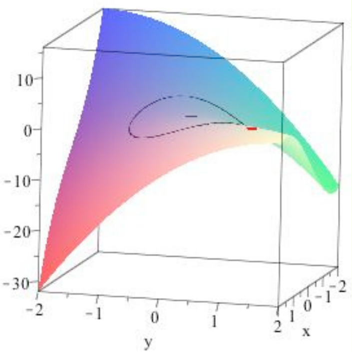
   - Critical points: $(0, 0)$ and $(4/3, 4/3)$.
   - $(0, 0)$ is a saddle point
   - $(4/3, 4/3)$ is a local maximum

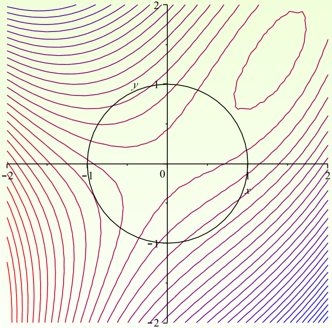
What if instead, I ask you to find the absolute maximum value $f(x, y)$ attains on the unit disk $\{(x, y) \mid x^2 + y^2 \leq 1\}$?

Absolute Extrema

Let $f(x, y) = 4xy - x^3 - 2y^2$.
Find the absolute maximum value $f(x, y)$ attains on the unit disk $\{(x, y) \mid x^2 + y^2 \leq 1\}$.

Know: Only Critical Points: $(0, 0)$ (saddle point); $(4/3, 4/3)$ (local max)

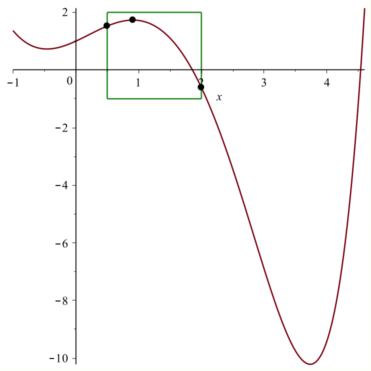




Notice: maximum value occurs on the boundary of the unit disk.

With a function of a single variable:

To find the absolute extrema of a function $f(x)$ on an interval $[a, b]$:



- Find critical points of f:
  Where is $f'(x) = 0$? Where d.n.e.?
- Relevant critical points?
  Critical points within (a, b).
- Determine absolute extrema:
  Evaluate $f(x)$ at each critical point, and $f(a)$, $f(b)$.
  - Largest value of $f =$ absolute maximum value on $[a, b]$
  - Smallest value of $f =$absolute minimum value on $[a, b]$

Absolute Extrema

Let $f(x, y) = 4xy - x^3 - 2y^2$.
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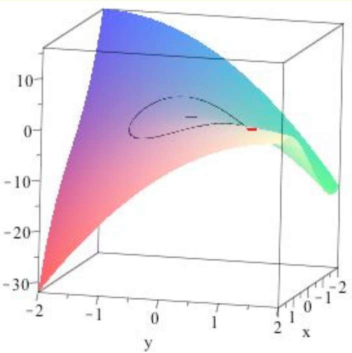

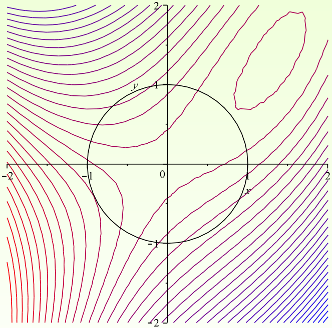

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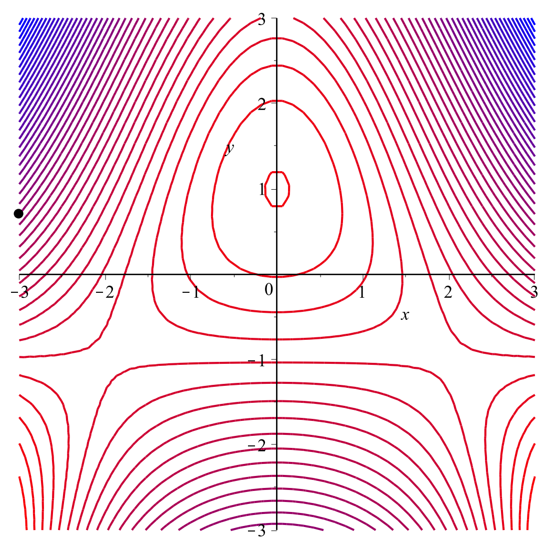
Where we're headed:

- We will look at how working with more than one variable affects how we approach finding absolute maxima and absolute minima.

- Applications of optimization are easy to think of.
- We will talk about a few less obvious ones:
  - The Method of Steepest Ascent (discuss only)
  - Linear Regression (more detail)

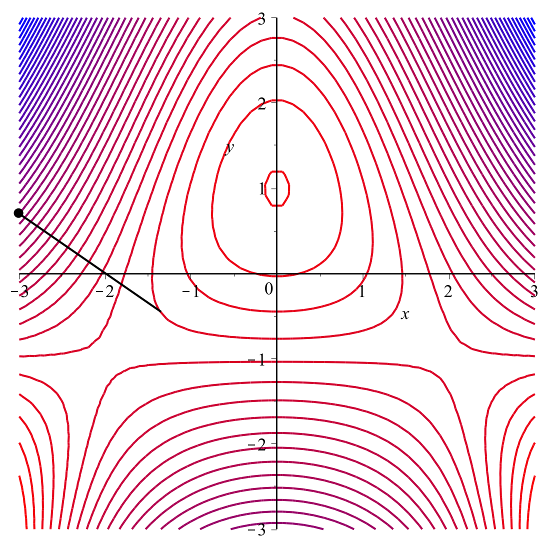
Method of Steepest Ascent

1. Pick any point that's a reasonable first guess for the local maximum



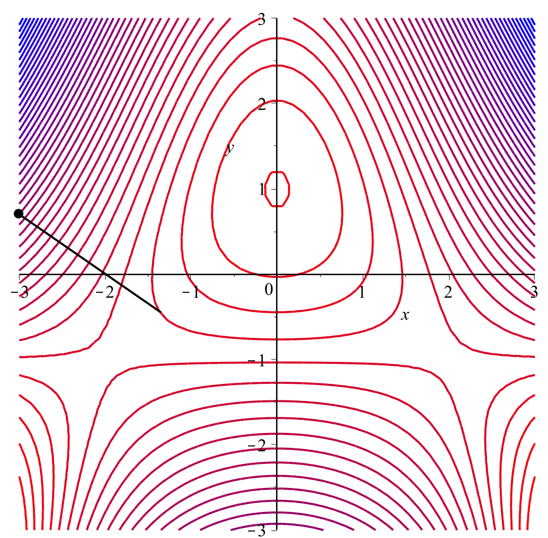
Method of Steepest Ascent

2. Move in the direction of steepest ascent (direction of the gradient).



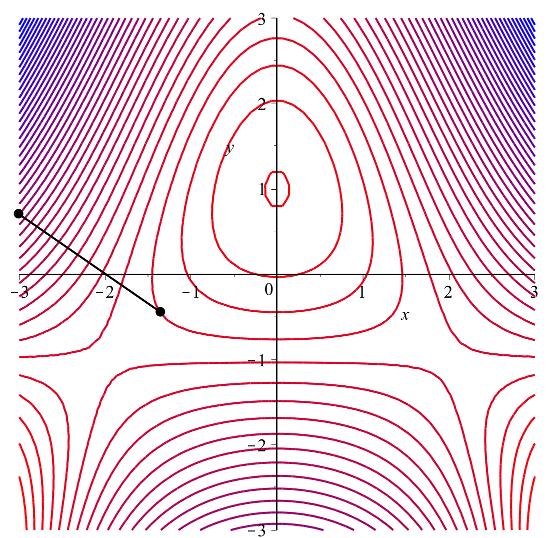
Method of Steepest Ascent

2. Keep moving in that direction until f is no longer increasing.



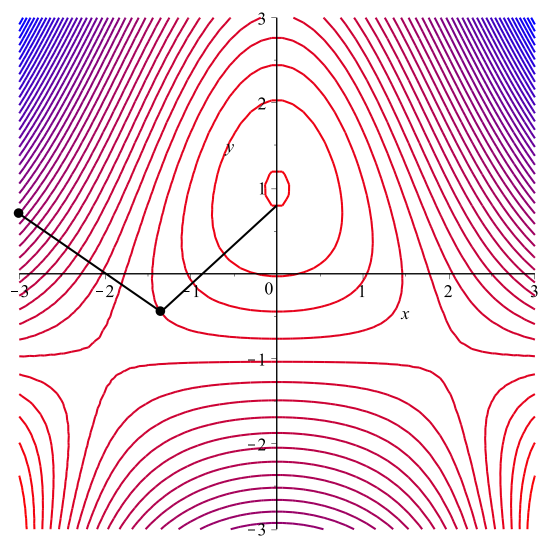
Method of Steepest Ascent

3. Stop. At new point, recalculate gradient.



Method of Steepest Ascent

2. Move in (new) direction of steepest ascent. Go until no longer going up.



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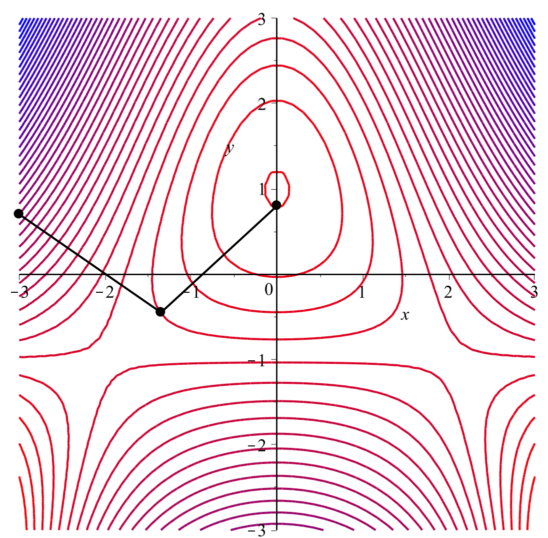

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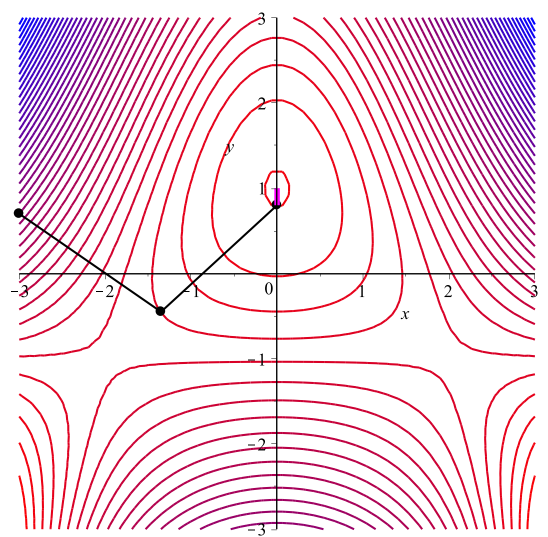

In Class Work

Let $f(x, y) = 4xy - x^3 - 2y^2$.

Already know: only critical points are $(0, 0)$ *(saddle point);* $(4/3, 4/3)$ *(local max).*

Find the absolute maximum value that $f(x, y)$ attains on the unit disk $\{(x, y) \mid x^2 + y^2 \leq 1\}$

Solutions

Let $f(x, y) = 4xy - x^3 - 2y^2$.

> *Already know: only critical points are* $(0, 0)$ *(saddle point);* $(4/3, 4/3)$ *(local max).*

Find the absolute maximum value that $f(x, y)$ attains on the unit disk $\{(x, y) \mid x^2 + y^2 \leq 1\}$

Since $(4/3, 4/3)$ is outside the disk, the absolute maximum value must occur **on the boundary of** the unit disk – that is, on the unit circle.

- How do we find where?

Solutions

Let $f(x, y) = 4xy - x^3 - 2y^2$.

> *Already know: only critical points are* $(0, 0)$ *(saddle point);* $(4/3, 4/3)$ *(local max).*

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- How do we find where?
- We know the boundary is parametrized by

$$x = \cos(t), y = \sin(t), z = f\big(\cos(t), \sin(t)\big).$$

Solutions

Let $f(x, y) = 4xy - x^3 - 2y^2$.

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- How do we find where?
- We know the boundary is parametrized by

$$x = \cos(t), y = \sin(t), z = f\big(\cos(t), \sin(t)\big).$$

- $\Longrightarrow$ We want to maximize $z = f\big(\cos(t), \sin(t)\big)$.

Solutions

Let $f(x,y) = 4xy - x^3 - 2y^2$.

Already know: only critical points are $(0,0)$ *(saddle point);* $(4/3, 4/3)$ *(local max).*

Find the absolute maximum value that $f(x,y)$ attains on the unit disk $\{(x,y) \mid x^2 + y^2 \leq 1\}$

- Max value occurs **on** unit circle. Maximize $g(t) = f\big(\overbrace{\cos(t)}^{x(t)}, \overbrace{\sin(t)}^{y(t)}\big)$.
- Find where $g'(t) = 0$! Need to find $g'(t)$!

$$\begin{aligned}
g'(t) &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\
&= (4y - 3x^2)(-\sin(t)) + (4x - 4y)(\cos(t)) \\
&= (4\sin(t) - 3(\cos(t))^2)(-\sin(t)) + (4\cos(t) - 4\sin(t))(\cos(t)) \\
&= -4\sin^2(t) + 3\cos^2(t)\sin(t) + 4\cos^2(t) - 4\cos(t)\sin(t) \\
&= 4\cos^2(t) - 4\sin^2(t) + 3\cos^2(t)\sin(t) - 4\sin(t)\cos(t)
\end{aligned}$$

Solutions

Let $f(x,y) = 4xy - x^3 - 2y^2$.

Already know: only critical points are $(0,0)$ *(saddle point);* $(4/3, 4/3)$ *(local max).*

Find the absolute maximum value that $f(x,y)$ attains on the unit disk $\{(x,y) \mid x^2 + y^2 \leq 1\}$

- Max value occurs **on** unit circle. Maximize $g(t) = f\big(\cos(t), \sin(t)\big)$.
- $g'(t) = 4\cos^2(t) - 4\sin^2(t) + 3\cos^2(t)\sin(t) - 4\sin(t)\cos(t)$. Find where $g'(t) = 0$
- Use Maple to solve for where $g'(t) = 0$ ... There are 4 critical points for $g(t)$ (That is, for critical points for $f(x,y)$ on the unit circle.) They are:

$$t \approx -.92 \qquad t \approx .68 \qquad t \approx 2.06 \qquad t \approx -2.71.$$

Solutions

Let $f(x, y) = 4xy - x^3 - 2y^2$.

> *Already know: only critical points are* $(0, 0)$ *(saddle point);* $(4/3, 4/3)$ *(local max).*

Find the absolute maximum value that $f(x, y)$ attains on the unit disk $\{(x, y) \mid x^2 + y^2 \leq 1\}$

- Max value occurs **on** unit circle. Maximize $g(t) = f\big(\cos(t), \sin(t)\big)$.
- $g'(t) = 4\cos^2(t) - 4\sin^2(t) + 3\cos^2(t)\sin(t) - 4\sin(t)\cos(t)$.
  Just found critical points on the boundary:

  $$t \approx -.92 \qquad t \approx .68 \qquad t \approx 2.06 \qquad t \approx -2.71.$$

- Plug each of these in (again, use Maple). ... Turns out f is largest at $t \approx -2.71$.
- But we need to know the x's and y's that give our max!
- Plug this value of t into $x = \cos(t)$, $y = \sin(t)$ ... Find that f attains its absolute max on the unit disk at $x \approx -.91$, $y \approx -.42$.

Solutions

Let $f(x,y) = 4xy - x^3 - 2y^2$.

Find the absolute maximum value that $f(x,y)$ attains on the unit disk $\{(x,y) \mid x^2 + y^2 \leq 1\}$

Found that maximum value of f on the unit disk occurs at approximately $(-.91, -.42)$.