Student Solutions to PS 2

Section 9.2

20. (Student answer)

Parametric equations for the position of an object are given. Find the object's velocity and speed, as well as describe the direction, at $t=0$, $t=\pi/2$.

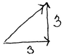
Position function: $\begin{cases} x(t) = 3\cos t + \sin 3t \\ y(t) = 3\sin t + \cos 3t \end{cases}$

$x'(t) = -3\sin t + 3\cos 3t$ $\qquad$ $y'(t) = 3\cos t - 3\sin 3t$

At $t=0$: horizontal component of velocity $= -3\sin 0 + 3\cos 0 = \boxed{3}$

vertical component of velocity $= 3\cos(0) - 3\sin(0) = \boxed{3}$

Speed $= \sqrt{3^2+3^2} = \sqrt{9+9} = \boxed{\sqrt{18}}$



At $t=0$, both components of velocity are positive $\Rightarrow$ the object is moving to the right and up

At $t=\frac{\pi}{2}$: horizontal component of velocity $= -3\sin\left(\frac{\pi}{2}\right) + 3\cos\left(\frac{3\pi}{2}\right) = \boxed{-3}$

vertical component of velocity $= +3\cos\left(\frac{\pi}{2}\right) - 3\sin\left(\frac{3\pi}{2}\right) = \boxed{3}$

speed $= \sqrt{(-3)^2+3^2} = \sqrt{18}$

At $t=\frac{\pi}{2}$, the horizontal component of velocity is negative while the vertical component is positive $\Rightarrow$ the object is moving to the left and up.

Student Solutions to PS 2

Section 9.4

12. (Student answer)

Find all polar coordinate representations of the rectangular point: $(-2, -\sqrt{5})$

$r^2 = x^2 + y^2$

$= (-2)^2 + (-\sqrt{5})^2$

$r = \pm\sqrt{4+5}$

$= \pm 3$

$\tan\theta = \frac{y}{x} = \frac{-\sqrt{5}}{-2}$

$\theta = \arctan\left(\frac{\sqrt{5}}{2}\right)$

≈ 0.841 radians

Since $(-2, -\sqrt{5})$ is in quadrant III, $\theta \approx 0.841$ rads corresponds to the negative value of r. The positive value of r corresponds to the angle that is π radians away from θ

The generic representation of the polar coordinates will be:

$\left(3, \arctan\left(\frac{\sqrt{5}}{2}\right) + \pi + 2\pi n\right)$ & $\left(-3, \arctan\left(\frac{\sqrt{5}}{2}\right) + 2\pi n\right)$,

for all integers n

Student Solutions to PS 2

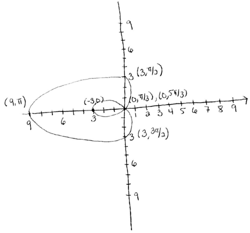
§9.4

36. (Student answer, amended a bit)

Sketch the graph of $r = 3 - 6\cos\theta$, and identify all values of θ where $r=0$ and a range of values of θ that produces one copy of the graph.

$r = 3 - 6\cos\theta$

θ	$3-6\cos\theta$
0	-3
$\pi/6$	~ -2.2
$\pi/3$	0
$\pi/2$	3
$2\pi/3$	6
$5\pi/6$	~ 8.2
π	9
$3\pi/2$	3
$5\pi/3$	0
2π	-3



$0 = 3 - 6\cos\theta$

$6\cos\theta = 3$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$ btwn 0 & 2π

$\Rightarrow r=0$ when $\theta = \frac{\pi}{3} + 2\pi n, \frac{5\pi}{3} + 2\pi n$, n any integer

Range of θ for one copy of the graph:

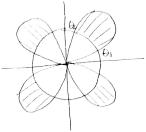
$0 \leq \theta \leq 2\pi$

Student Solutions to PS 2

Section 9.5

(Student answer)

28. Find area inside $r_1 = 2\sin(2\theta)$ and outside $r_2 = 1$



How to:

- Find intersections
- Use intersections to subtract $\int_{\theta_1}^{\theta_2} \frac{1}{2}(r_2)^2 d\theta$ from $\int_{\theta_1}^{\theta_2} \frac{1}{2}(r_1)^2 d\theta$
- Multiply first portion by 4 - should all have equal areas

Find intersections

$r_1 = r_2$

$2\sin(2\theta) = 1$

$\sin(2\theta) = \frac{1}{2}$

True for $2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ (these are all we need because all four areas are the same)

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$

$\theta_1 = \frac{\pi}{12}, \theta_2 = \frac{5\pi}{12}$

Find area of one region:

$$\frac{1}{4}(\text{Area}) = \int_{\theta_1}^{\theta_2} \frac{1}{2}(r_1)^2 - \frac{1}{2}(r_2)^2 d\theta = \int_{\pi/12}^{5\pi/12} \frac{1}{2}(2\sin 2\theta)^2 - \frac{1}{2}(1)^2 d\theta$$

$$= \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \frac{1}{2}(4\sin^2 2\theta) - \frac{1}{2} d\theta = \int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} 2\left(\frac{1-\cos 4\theta}{2}\right) d\theta - \left[\frac{\theta}{2}\right]_{\pi/12}^{5\pi/12}$$

$$= \left[\theta - \frac{1}{4}\sin 4\theta\right]_{\pi/12}^{5\pi/12} - \left[\frac{5\pi}{24} - \frac{\pi}{24}\right] = \left[\left(\frac{5\pi}{12} - \frac{1}{4}\cdot\left(-\frac{\sqrt{3}}{2}\right)\right) - \left(\frac{\pi}{12} - \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)\right)\right] - \left[\frac{4\pi}{24}\right]$$

$$= \frac{4\pi}{12} + \frac{\sqrt{3}}{4} - \frac{4\pi}{24} = \frac{\pi}{6} + \frac{\sqrt{3}}{4}$$

Find total area:

$$\text{Area} = 4\left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right) = \boxed{\frac{2\pi}{3} + \sqrt{3}}$$

Formal Definition of the Cross Product

For two vectors $\overrightarrow{\mathbf{a}} = a_1\,\overrightarrow{\mathbf{i}} + a_2\,\overrightarrow{\mathbf{j}} + a_3\,\overrightarrow{\mathbf{k}}$ and $\overrightarrow{\mathbf{b}} = b_1\,\overrightarrow{\mathbf{i}} + b_2\,\overrightarrow{\mathbf{j}} + b_3\,\overrightarrow{\mathbf{k}}$, we define the **cross product**, or **vector product**, of $\overrightarrow{\mathbf{a}}$ and $\rightarrow$ to be:

$$\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}} = (a_2 b_3 - a_3 b_2)\,\overrightarrow{\mathbf{i}} - (a_1 b_3 - a_3 b_1)\,\overrightarrow{\mathbf{j}} + (a_1 b_2 - a_2 b_1)\,\overrightarrow{\mathbf{k}}$$

In Class Work

Let $\overrightarrow{\mathbf{a}} =< 1, -1, 2 >$ and $\overrightarrow{\mathbf{b}} =< -3, 0, 1 >$.

1. Find $\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}}$.
2. Find $\overrightarrow{\mathbf{a}}(\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}})$, $\overrightarrow{\mathbf{b}} \cdot (\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}})$. What do you conclude the relationship between $\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}}$ and $\overrightarrow{\mathbf{a}}$, $\overrightarrow{\mathbf{b}}$?
3. Find $\overrightarrow{\mathbf{b}} \times \overrightarrow{\mathbf{a}}$. Is the cross product a commutative operation?

Solutions

Let $\overrightarrow{\mathbf{a}} =< 1, -1, 2 >$ and $\overrightarrow{\mathbf{b}} =< -3, 0, 1 >$.

1. Find $\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}}$.

$$
\begin{aligned}
\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}} &= \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} \overrightarrow{\mathbf{i}} \boxed{-} \begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix} \overrightarrow{\mathbf{j}} + \begin{vmatrix} 1 & -1 \\ -3 & 0 \end{vmatrix} \overrightarrow{\mathbf{k}} \\
&= (-1 - 0)\overrightarrow{\mathbf{i}} - (1 + 6)\overrightarrow{\mathbf{j}} + (0 - 3)\overrightarrow{\mathbf{k}} \\
&= < -1, -7, -3 >
\end{aligned}
$$

Solutions

Let $\overrightarrow{\mathbf{a}} =< 1, -1, 2 >$ and $\overrightarrow{\mathbf{b}} =< -3, 0, 1 >$.
In (1), found that $\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}} =< -1, -7, -3 >$

2. Check that $\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}}$ is in fact orthogonal to both $\overrightarrow{\mathbf{a}}$ and $\overrightarrow{\mathbf{b}}$.

$$\begin{aligned} < 1, -1, 2 > \cdot < -1, -7, -3 > &= -1 + 7 - 6 = 0 \\ < -3, 0, 1 > \cdot < -1, -7, -3 > &= 3 + 0 - 3 = 0 \end{aligned}$$

Since $\overrightarrow{\mathbf{a}} \cdot (\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}}) = 0$ and $\overrightarrow{\mathbf{b}} \cdot (\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}}) = 0$, $\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}}$ is orthogonal to both $\overrightarrow{\mathbf{a}}$ and $\overrightarrow{\mathbf{b}}$.

Solutions

Let $\overrightarrow{\mathbf{a}} =< 1, -1, 2 >$ and $\overrightarrow{\mathbf{b}} =< -3, 0, 1 >$.
In (1), found that $\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}} =< -1, -7, -3 >$

3. Find $\overrightarrow{\mathbf{b}} \times \overrightarrow{\mathbf{a}}$. Is the cross product a commutative operation?

$$\begin{aligned}
\overrightarrow{\mathbf{b}} \times \overrightarrow{\mathbf{a}} &= \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} \overrightarrow{\mathbf{i}} - \begin{vmatrix} -3 & 1 \\ 1 & 2 \end{vmatrix} \overrightarrow{\mathbf{j}} + \begin{vmatrix} -3 & 0 \\ 1 & -1 \end{vmatrix} \overrightarrow{\mathbf{k}} \\
&= (0+1)\overrightarrow{\mathbf{i}} - (-6-1)\overrightarrow{\mathbf{j}} + (3-0)\overrightarrow{\mathbf{k}} \\
&= < 1, 7, 3 > \\
&= -\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}}
\end{aligned}$$

The cross product is in fact **anti-commutative** – $\overrightarrow{\mathbf{b}} \times \overrightarrow{\mathbf{a}}$ will **always** be in the opposite direction as $\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}}$.

Is $\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}}$ always orthogonal to $\overrightarrow{\mathbf{a}}$ and $\overrightarrow{\mathbf{b}}$?

Let

$$\overrightarrow{\mathbf{a}} = a_1 \overrightarrow{\mathbf{i}} + a_2 \overrightarrow{\mathbf{j}} + a_3 \overrightarrow{\mathbf{k}} \qquad \overrightarrow{\mathbf{b}} = b_1 \overrightarrow{\mathbf{i}} + b_2 \overrightarrow{\mathbf{j}} + b_3 \overrightarrow{\mathbf{k}}$$

Recall: $\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}} = (a_2 b_3 - a_3 b_2)\overrightarrow{\mathbf{i}} - (a_1 b_3 - a_3 b_1)\overrightarrow{\mathbf{j}} + (a_1 b_2 - a_2 b_1)\overrightarrow{\mathbf{k}}$

$$\begin{aligned}
\overrightarrow{\mathbf{a}} \cdot (\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}}) &= (a_1 \overrightarrow{\mathbf{i}} + a_2 \overrightarrow{\mathbf{j}} + a_3 \overrightarrow{\mathbf{k}}) \\
&\quad \cdot ((a_2 b_3 - a_3 b_2)\overrightarrow{\mathbf{i}} - (a_1 b_3 - a_3 b_1)\overrightarrow{\mathbf{j}} + (a_1 b_2 - a_2 b_1)\overrightarrow{\mathbf{k}} \\
&= a_1(a_2 b_3 - a_3 b_2) - a_2(a_1 b_3 - a_3 b_1) + a_3(a_1 b_2 - a_2 b_1) \\
&= a_1 a_2 b_3 - a_1 a_3 b_2 - a_1 a_2 b_3 + a_2 a_3 b_1 + a_1 a_3 b_2 - a_2 a_3 b_1 \\
&= 0
\end{aligned}$$

Similarly, $\overrightarrow{\mathbf{a}} \cdot (\overrightarrow{\mathbf{b}} \times \overrightarrow{\mathbf{a}}) = 0$, $\overrightarrow{\mathbf{b}} \cdot (\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}}) = 0$, and $\overrightarrow{\mathbf{b}} \cdot (\overrightarrow{\mathbf{b}} \times \overrightarrow{\mathbf{a}}) = 0$.