Overview of §12.4:

- If $f(x)$ is a differentiable function of one variable, we can use the tangent line at $x = x_0$,

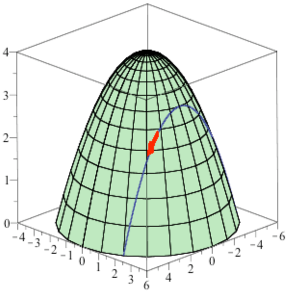
  $$y = L(x) = f'(x_0)(x - x_0) + f(x_0)$$

  as a **linear approximation** of f at x_0.

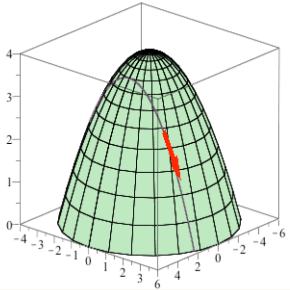
  That is, we can use $L(x)$ to approximate $f(x)$ at points near x_0.

- In the same way, for a function $f(x, y)$, the *tangent plane* at (a, b) will give a linear approximation of f at (a, b).

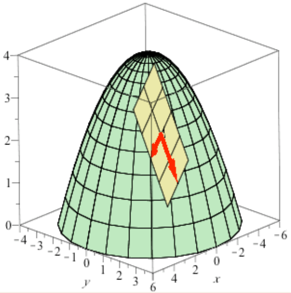
What IS a tangent plane?



Above, the red arrow represents the line to the curve formed by the intersection of the plane $y = b$ with the surface $z = f(x, y)$; that is, it is tangent to $z = f(x, b)$

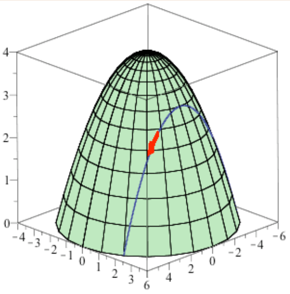


This arrow represents the tangent line to the curve at (a, b) when $x = a$ is fixed; that is, it is tangent to $z = f(a, y)$.

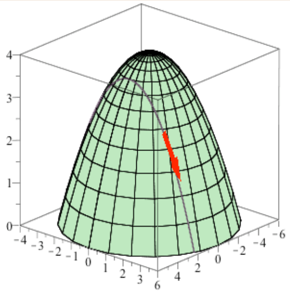


These two lines (and every other tangent line to a curve on the surface that goes through the point $(a, b, f(a, b))$ will lie on our **tangent plane**, shown in yellow.

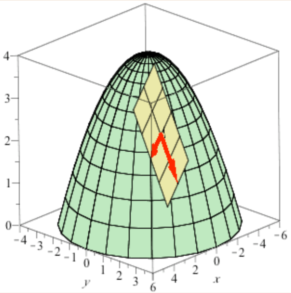
Tangent Plane:



The tangent plane must contain the line tangent to $z = f(x, b)$



It must also contain the line tangent to $z = f(a, y)$.



We can find the normal vector to the tangent plane by taking the cross product of these

When does a tangent plane at (a, b) exist?

- In order for a tangent **line** to exist, all we need is for $f'(a)$ to exist. If it does, we say f is **differentiable** at $x = a$.

- Just because you can find the equation of the plane

$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

does NOT automatically make it tangent to the surface!

As we've discussed, in the two dimensional case, there are an infinite number of tangent lines at a point. In order for the concept of tangent plane to make sense, we need *all* of those tangent lines to lie in the same plane.

A function that meets that condition will be **differentiable**.

Differentiability in a single-variable function:

Definition of $f'(a)$:

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

In other words,

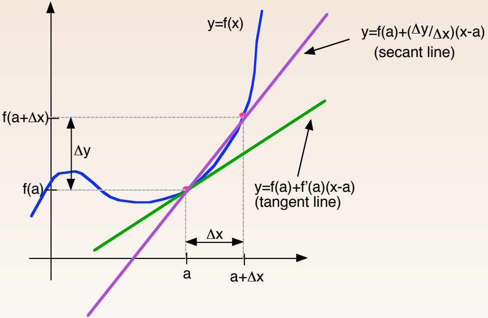
$$\frac{\Delta y}{\Delta x} \to f'(a) \text{ as } \Delta x \to 0.$$

Let $\epsilon = \dfrac{\Delta y}{\Delta x} - f'(a)$.

- $\Delta y = f'(a)\Delta x + \epsilon \Delta x$
- $\epsilon \to 0$ as $\Delta x \to 0$

Thinking about differentiability in a single-variable function:

Graphically:



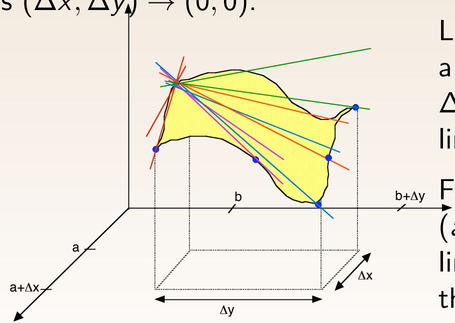
Let $\epsilon = \dfrac{\Delta y}{\Delta x} - f'(a) =$ the difference in the slope of the secant line on $[a, a + \Delta x]$ and the tangent line. Notice that $\epsilon \to 0$ as $\Delta x \to 0$.

Differentiability of a multivariate function:

A function $f(x, y)$ is **differentiable** at (a, b) if and only if we can write

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

where ϵ_1 and ϵ_2 are both functions of Δx and Δy, and $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.



Lines of the same color represent a secant line on the square $[a, a+\Delta x] \times [b, b+\Delta y]$ and a tangent line in same xy direction.

For $f(x, y)$ to be differentiable at (a, b), the slope of each secant line on this square must approach the slope of the corr. tangent line.

Fortunately, we are able to only use two ϵs, ϵ_1 and ϵ_2.

In Class Work

Let $f(x, y) = e^{-x^2-y^2}$.

- Verify that at the point $\left(1, -2, e^{-5}\right)$, a tangent plane exists
- Find the equation of that plane.

Solutions

Let $f(x,y) = e^{-x^2-y^2}$.

- Verify that at the point $\left(1, -2, e^{-5}\right)$, a tangent plane exists
  - $f_x(x,y) = -2xe^{-x^2-y^2}$ and $f_y(x,y) = -2ye^{-x^2-y^2}$
  - Both are continuous at $(1,-2)$ (in fact, everywhere), so by theorem, tangent plane at $(1,-2)$ exists
- Find the equation of that plane.
  - Normal vector to plane tangent at $x = 1$, $y = -2$:

$$< f_x(1,-2), f_y(1,-2), -1 > .$$

  - Thus at $(1, -2, e^{-5})$,

$$\Rightarrow \overrightarrow{\mathbf{n}} =< -2e^{-5}, 4e^{-5}, -1 >,$$

  - Hence the tangent plane is given by the equation

$$-2e^{-5}(x-1) + 4e^{-5}(y+2) - 1(z - e^{-5}) = 0$$

or

$$z = -2e^{-5}(x-1) + 4e^{-5}(y+2) + e^{-5}.$$