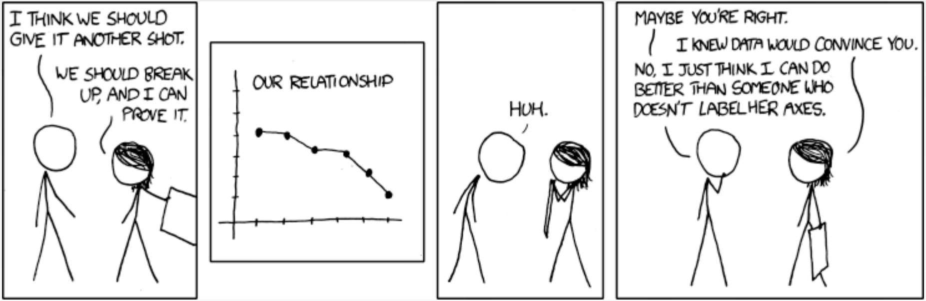
Label Your Axes!



Recall from last time:

Consider a function $f(x, y)$ and a point $\big(a, b, f(a, b)\big)$.

- If $f(x, y)$ has continuous partials f_x and $f_y$on a neighborhood surrounding (a, b), then f is differentiable at (a, b).
- If $f(x, y)$ is differentiable at (a, b), then the surface $z = f(x, y)$ has a *non-vertical* tangent plane at the point $\big(a, b, f(a, b)\big)$
- If such a non-vertical tangent plane exists,
  - Since tangent lines parallel to x and y axes lie in tangent plane, can find normal vector to the tangent plane by evaluating $< 0, 1, f_y(a, b) > \times < 1, 0, f_x(a, b) >$. Turns out to be:

    $$\overrightarrow{\mathbf{n}} =< f_x(a, b), f_y(a, b), -1 > .$$

  - Since $\big(a, b, f(a, b)\big)$, the point of tangency, lies on the tangent plane, equation for the plane tangent to $z = f(x, y)$ at $\big(a, b, f(a, b)\big)$ is

    $$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$

Example :

Let $f(x, y) = xy^2$.

- Can we use our formula to find the tangent plane?
  - $f_x(x, y) = y^2$ continuous everywhere
  - $f_y(x, y) = 2xy$ continuous everywhere
  - Thus $f(x, y)$ is differentiable at (a, b), and the equation of the tangent plane at (a, b) is given by

$$z = f_x(a, b)(x - a) + f_y(y - b) + f(a, b)$$

- Find the equation of the plane tangent to $z = f(x, y)$ at the point $(2, 3, 18)$.

$$f_x(2, 3) = 3^2 = 9 \qquad f_y(2, 3) = 2(2)(3) = 12.$$

Thus the tangent plane at the point $(2, 3, 18)$ is

$$z = 9(x - 2) + 12(y - 3) + 18.$$

Question:

Let $L(x, y)$ be the linear approximation of $f(x, y)$ at (a, b).

What graphical properties of the surface $f(x, y)$ would make $L(x, y)$ be:

- particularly accurate?
- particularly inaccurate?

Another Question:

If $f(x, y)$ is a well-behaved function and has a local maximum at (a, b), what can you say about the linear approximation to $f(x, y)$ at (a, b)?

In Class Work

Let $f(x, y) = e^{-x^2-y^2}$.

Last time, you found that the plane tangent to $z = f(x, y)$ at the point $(1, -2, e^{-5})$ has equation

$$z = -2e^{-5}(x-1) + 4e^{-5}(y+2) + e^{-5}.$$

1. Use a linear approximation of $f(x, y)$ to approximate f at $(0.96, -1.97)$.

Solutions

Let $f(x,y) = e^{-x^2-y^2}$.

1. Use a linear approximation of $f(x,y)$ to approximate f at $(0.96, -1.97)$.

A good linear approximation of $f(x,y)$ to use to approximate f at $(0.96, -1.97)$ is the linear approximation at $(1,-2)$. Since the linear approximation is the same as the tangent plane, we already have the linear approximation we need:

$$L(x,y) = -2e^{-5}(x-1) + 4e^{-5}(y+2) + e^{-5}.$$

We can use this to approximate $f(0.96, -1.97)$:

$$\begin{aligned} f(0.96,-1.97) &\approx L(0.96,-1.97) = -2e^{-5}(0.96-1) + 4e^{-5}(-1.97-(-2)) \\ &= 0.08e^{-5} + 0.12e^{-5} + e^{-5} = 1.2e^{-5} \approx 0.00809. \end{aligned}$$

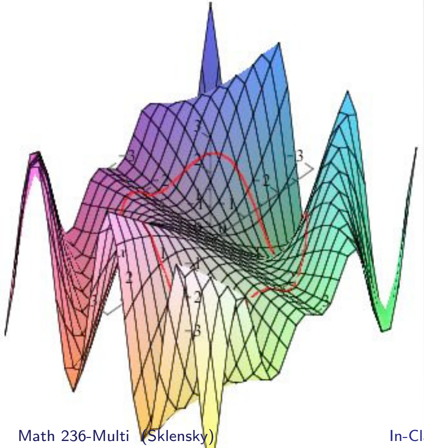
For comparison purposes, from Maple I find that $f(0.96,-1.97) \approx 0.0082$.

Motivating the Chain Rule

Suppose

- $f(x, y) = x\cos(xy)$ models surface of a mountain
- $r(t) =< 2\cos(t), 2\sin(t) >$ models xy-coordinates of a circular path on the mountain.

Then $f \circ r(t)$ gives the altitude of any point on that circular path.



Question: How is the elevation changing at $t = \frac{\pi}{2}$?