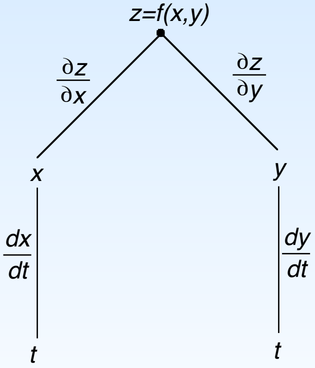
Recall:

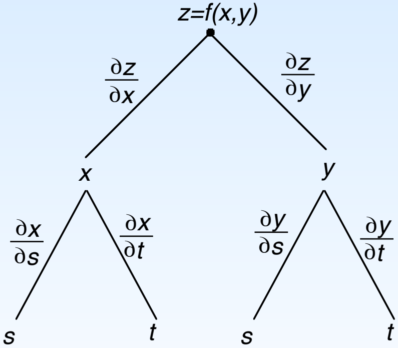
Given $f(x, y)$, where x, y are functions of t



$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Also recall:

Given $f(x, y)$, where x, y are both functions of s, t



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

In Class Work

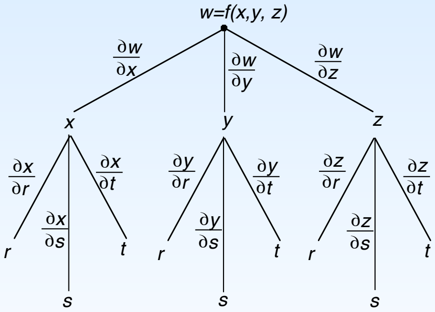
1. Suppose that $w = f(x, y, z)$ and that x, y, and z are all functions of r, s, and t.
   (a) How many partial derivatives do you need to calculate, in order to determine $\frac{\partial w}{\partial t}$?
   (b) What *is* the expression for $\frac{\partial w}{\partial t}$?

2. Draw a tree diagram to figure out the chain rule for composite functions of the form $z = g(u, v) = f(x(r, s), y(r, s, t))$, where r, s, and t are all functions of u and v.

3. Use the chain rule twice to find $g''(t)$ if $g(t) = f(x(t), y(t))$.

Solutions

1.How many partial derivatives do you need to calculate, in order to determine $\frac{\partial w}{\partial t}$?

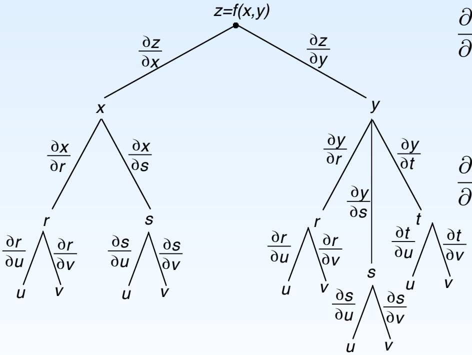


Just to calculate this one partial using the chain rule, I need to calculate six different partial derivatives.

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

Solutions

2. Draw a tree diagram to figure out the chain rule for composite functions of the form $z = g(u, v) = f(x(r, s), y(r, s, t))$, where r, s, and t are all functions of u and v.



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial r}\frac{\partial r}{\partial u} + \frac{\partial x}{\partial s}\frac{\partial s}{\partial u}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial r}\frac{\partial r}{\partial u} + \frac{\partial y}{\partial s}\frac{\partial s}{\partial u} + \frac{\partial y}{\partial t}\frac{\partial t}{\partial u}\right)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial r}\frac{\partial r}{\partial v} + \frac{\partial x}{\partial s}\frac{\partial s}{\partial v}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial r}\frac{\partial r}{\partial v} + \frac{\partial y}{\partial s}\frac{\partial s}{\partial v} + \frac{\partial y}{\partial t}\frac{\partial t}{\partial v}\right)$$

Solutions

3. Use the chain rule twice to find $g''(t)$ if $g(t) = f(x(t), y(t))$.

$$g'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \Longrightarrow g''(t) = \frac{d}{dt}\left(\frac{\partial f}{\partial x}\frac{dx}{dt}\right) + \frac{d}{dt}\left(\frac{\partial f}{\partial y}\frac{dy}{dt}\right).$$

Use the product rule on both products in the sum on the right:

$$g''(t) = \left[\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{dx}{dt} + \frac{\partial f}{\partial x}\cdot\frac{d^2x}{dt^2}\right] + \left[\frac{d}{dt}\left(\frac{\partial f}{\partial y}\right)\cdot\frac{dy}{dt} + \frac{\partial f}{\partial y}\cdot\frac{d^2y}{dt^2}\right]$$

Apply the chain rule to both $\frac{\partial f}{\partial x}(x(t), y(t))$ and $\frac{\partial f}{\partial y}(x(t), y(t))$:

$$\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y \partial x}\frac{dy}{dt} \qquad \frac{d}{dt}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y^2}\frac{dy}{dt}.$$

So
$$g''(t) = \frac{\partial^2 f}{\partial x^2}\left(\frac{dx}{dt}\right)^2 + \frac{\partial^2 f}{\partial y \partial x}\left(\frac{dy}{dt}\right)\left(\frac{dx}{dt}\right) + \frac{\partial f}{\partial x}\frac{d^2x}{dt^2}$$
$$+ \frac{\partial^2 f}{\partial x \partial y}\left(\frac{dx}{dt}\right)\left(\frac{dy}{dt}\right) + \frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dt}\right)^2 + \frac{\partial f}{\partial y}\frac{d^2y}{dt^2}$$

Recall:

If we have $z = f(x, y)$, then the equation of the tangent plane at (a, b) (when $f(x, y)$ has continuous first partials at (a, b)) is

$$z = f_x(a, b)(x - a) + f_y(y - b) + f(a, b).$$

Note: This equation only gives the tangent plane if the tangent plane is non-vertical.

Rephrasing this slightly:

If we have $z = f(x, y)$, then the equation of the tangent plane at the point (a, b, c) (when $f(x, y)$ has continuous partials at $x = a$, $y = b$) is

$$z = z_x(a, b)(x - a) + z_y(a, b)(y - b) + c.$$