Definition: A subgroup H of a group G is a **normal subgroup** iff $aH = Ha$ for all $a \in G$. We denote a normal subgroup by $H \lhd G$.

Theorem 9.1: A subgroup $H \ \leq \ G$ is normal $\Longleftrightarrow$ $(x)^{-1}Hx \subseteq H$ for all $x \in G$.

November 13, 2002

Let $H = \{\epsilon, \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}\}$. We've seen that $H \triangleleft S_3$. We let

$$\begin{aligned} S_H &= \text{ the set of all left cosets of } H \text{ in } S_3 \\ &= \{\alpha H | \alpha \in S_3\} \\ &= \{H, \begin{bmatrix} 1 & 2 \end{bmatrix} H\}, \end{aligned}$$

and we defined an operation $*$ on the elements of the set S_H as follows: For any $\alpha H, \beta H \in S_H$,

$$\alpha H * \beta H \overset{def}{=} (\alpha \circ \beta) H.$$

We saw that if $\alpha_1 H = \alpha_2 H$ and $\beta_1 H = \beta_2 H$, then $\alpha_1 H * \beta_1 H = \alpha_2 H * \beta_2 H$. This shows that the operation $*$ is *well-defined*.

Let $K = \{\epsilon, \begin{bmatrix}1 & 2\end{bmatrix}\}$. We've seen that $K \not\triangleleft S_3$.

We let

$$\begin{aligned} S_K = &\text{ the set of all left cosets of } K \text{ in } S_3 \\ &= \{\alpha K | \alpha \in S_3\} \\ &= \{K, \begin{bmatrix}1 & 3\end{bmatrix} K, \begin{bmatrix}2 & 3\end{bmatrix} K\}. \end{aligned}$$

If we use the same idea as before to define an operation $*$ on the elements of the set S_K: For any $\alpha K, \beta K \in S_K$, define

$$\alpha K * \beta K \overset{def}{=} (\alpha \circ \beta) K,$$

then it turns out that $*$ is *not* well-defined on S_K. For although

$$\begin{bmatrix}1 & 3\end{bmatrix} K = \begin{bmatrix}1 & 2 & 3\end{bmatrix} K \text{ and } \begin{bmatrix}2 & 3\end{bmatrix} K = \begin{bmatrix}1 & 3 & 2\end{bmatrix} K,$$

it is not true that
$\begin{bmatrix}1 & 3\end{bmatrix} K * \begin{bmatrix}2 & 3\end{bmatrix} K = \begin{bmatrix}1 & 2 & 3\end{bmatrix} K * \begin{bmatrix}1 & 3 & 2\end{bmatrix} K.$