The 8 motions on the square

$$\{R_0, R_{90}, R_{180}, R_{270}, H, D, V, D'\}$$

and the operation $\circ$ of combining the motions form a system called the **dihedral group of order 8**, denoted $D_4$.

Below is the *Cayley* table showing the result of the applying the operation to any 2 elements.

| $\circ$ | $R_0$ | $R_{90}$ | $R_{180}$ | $R_{270}$ | $H$ | $D$ | $V$ | $D'$ |
|---|---|---|---|---|---|---|---|---|
| $R_0$ | $R_0$ | $R_{90}$ | $R_{180}$ | $R_{270}$ | $H$ | $D$ | $V$ | $D'$ |
| $R_{90}$ | $R_{90}$ | $R_{180}$ | $R_{270}$ | $R_0$ | $D'$ | $H$ | $D$ | $V$ |
| $R_{180}$ | $R_{180}$ | $R_{270}$ | $R_0$ | $R_{90}$ | $V$ | $D'$ | $H$ | $D$ |
| $R_{270}$ | $R_{270}$ | $R_0$ | $R_{90}$ | $R_{180}$ | $D$ | $V$ | $D'$ | $H$ |
| $H$ | $H$ | $D$ | $V$ | $D'$ | $R_0$ | $R_{90}$ | $R_{180}$ | $R_{270}$ |
| $D$ | $D$ | $V$ | $D'$ | $H$ | $R_{270}$ | $R_0$ | $R_{90}$ | $R_{180}$ |
| $V$ | $V$ | $D'$ | $H$ | $D$ | $R_{180}$ | $R_{270}$ | $R_0$ | $R_{90}$ |
| $D'$ | $D'$ | $H$ | $D$ | $V$ | $R_{90}$ | $R_{180}$ | $R_{270}$ | $R_0$ |

# Important Aspects of the Cayley Table for $D_4$

1. *Closure:* No new motions are introduced. If $A, B \in D_4$, then $A \circ B \in D_4$.

2. *Identity:* $R_0$ acts as an identity motion — $R_0 \circ A = A \circ R_0 = A$ for all $A \in D_4$.

3. *Inverse:* Every element has an inverse motion that "undoes" what the motion does. For example, $R_{90} \circ R_{270} = R_{270} \circ R_{90} = R_0$.

4. *Associativity:* You can check the table that $(A \circ B) \circ C = A \circ (B \circ C)$ for all $A, B, C \in D_4$. This makes sense if you think of the motions as functions, and one motion followed by another as composition.

5. The motions are not commutative. For example, $R_{90} \circ H = D$ but $H \circ R_{90} = D'$. But some motions do commute: $R_{90} \circ R_{180} = R_{180} \circ R_{90} = R_{270}$.

6. Although it's not obvious from the table, all of the motions can be obtained by some combination of $R_{90}$ and $H$. In this sense, these two motions **generate** $D_4$.

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