Thm 6.2: Properties of Isomorphisms Acting on Elements

Suppose that $\phi : G \to \bar{G}$ is an isomorphism. Then

1. ϕ carries the identity of G to the identity of $\bar{G}$.
2. For every integer n and for every group elt $a \in G$, $\phi(a^n) = [\phi(a)]^n$.
3. For any elements $a, b \in G$, $ab = ba \Leftrightarrow \phi(a)\phi(b) = \phi(b)\phi(a)$.
4. $G =< a >$ if and only if $\bar{G} =< \phi(a) >$.
5. $|a| = |\phi(a)|$ for all a in G. (That is, isomorphisms preserve order).
6. For a fixed $k \in \mathbb{Z}$ and a fixed $b \in G$, the eqn $x^k = b$ has the same number of solutions in G as does the eqn $x^k = \phi(b)$ in $\bar{G}$.
7. If G is finite, then G and $\bar{G}$ have exactly the same number of elements of every order.

Thm 6.3: Properties of Isomorphisms Acting on Groups

Suppose that $\phi : G \to \bar{G}$ is an isomorphism. Then

1. ϕ^{-1} is an isomorphism from $\bar{G}$ to G.
2. G is Abelian if and only if $\bar{G}$ is Abelian.
3. G is cyclic if and only if $\bar{G}$ is cyclic.
4. If K is a subgroup of G, then $\phi(K) = \{\phi(k) | k \in K\}$ is a subgroup of $\bar{G}$.

In Class Work

Show that $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_8$, but that $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_6$.

Solutions

Show that $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_8$, but that $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_6$.

To show $\mathbb{Z}_\mathbf{2} \oplus \mathbb{Z}_\mathbf{4} \not\approx \mathbb{Z}_\mathbf{8}$:

$$\begin{aligned} \mathbb{Z}_8 &= \{0,1,2,3,4,5,6,7\} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_4 &= \{(0,0),(1,0),(0,1),(1,1),(0,2), \\ &\qquad (1,2),(0,3),(1,3)\} \end{aligned}$$

$$\begin{array}{llll} \big|(0,0)\big| = 1 & \big|(1,0)\big| = 2 & \big|(0,1)\big| = 4 & \big|(1,1)\big| = 4 \\ \big|(0,2)\big| = 2 & \big|(1,2)\big| = 2 & \big|(0,3)\big| = 4 & \big|(1,3)\big| = 4 \end{array}$$

Because no element in $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ has order 8, $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ is not cyclic. Since $\mathbb{Z}_8$ *is* cyclic, this means that there can not be any isomorphism between the two groups (Theorem 6.3, Part 2).

Solutions:

Show that $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_8$, but that $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_6$.

Is it true that $\mathbb{Z}_\mathbf{2} \oplus \mathbb{Z}_\mathbf{3} \approx \mathbb{Z}_\mathbf{6}$?

$$\begin{aligned} \mathbb{Z}_6 &= \{0, 1, 2, 3, 4, 5\} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_3 &= \{(0,0), (1,0), (0,1), (1,1), (0,2), (1,2), \} \end{aligned}$$

In $\mathbb{Z}_2 \oplus \mathbb{Z}_3$, the order of $(1,1)$ is 6:

$$(1,1) + (1,1) + \dots (1,1) = (k \cdot 1 \mod 2, k \cdot 1 \mod 3),$$

and the first time we'll get 0 in both components is when k=6.)

Thus we know that both groups are cylic of order 6. Does that mean they're isomorphic?

Solutions:

Show that $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_8$, but that $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_6$.

To **show** that $\mathbb{Z}_\mathbf{2} \oplus \mathbb{Z}_\mathbf{3} \approx \mathbb{Z}_\mathbf{6}$, we need to construct an isomorphism from $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ to $\mathbb{Z}_6$ or vice versa.

First, just try to define a function, then see if it's an isomorphism.

Elements in $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ have the form (x, y), with $x \in \mathbb{Z}_2$ and $y \in Z_3$.

To define a function $f : \mathbb{Z}_6 \to \mathbb{Z}_2 \oplus \mathbb{Z}_3$, $f(m)$ has to be something of the form (x, y), where x and y have to either just be something basic in the group like 0 or 1, or be something that can be gotten from m in some way.

$m \in \mathbb{Z}_6$, and we need x to be in $\mathbb{Z}_2$ and y to be in $\mathbb{Z}_3$.

Solutions

Show that $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_8$, but that $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_6$.

To **show** that $\mathbb{Z}_\mathbf{2} \oplus \mathbb{Z}_\mathbf{3} \approx \mathbb{Z}_\mathbf{6}$, we need to construct an isomorphism.

Try defining $f : \mathbb{Z}_6 \to \mathbb{Z}_2 \oplus \mathbb{Z}_3$ by $f(m) = (m \mod 2, m \mod 3)$.

Solutions

Define $f : \mathbb{Z}_6 \to \mathbb{Z}_2 \oplus \mathbb{Z}_3$ by $f(m) = (m \mod 2, m \mod 3)$.

Is f well-defined?

$$\begin{aligned} a = b \mod 6 &\implies 6 \big| (a-b) \\ &\implies 2 \big| (a-b) \text{ and } 3 \big| (a-b) \\ &\implies a = b \mod 2 \text{ and } a = b \mod 3 \\ &\implies (a \mod 2, a \mod 3) \\ &\qquad = (b \mod 2, b \mod 3) \end{aligned}$$

Thus $a = b \implies f(a) = f(b)$, so f is well-defined.

Solutions

Define $f : \mathbb{Z}_6 \to \mathbb{Z}_2 \oplus \mathbb{Z}_3$ by $f(m) = (m \mod 2, m \mod 3)$.

Is f 1-1?

$$\begin{aligned} f(a) = f(b) &\implies (a \mod 2, a \mod 3) = (b \mod 2, b \mod 3) \\ &\implies a = b \mod 2 \text{ and } a = b \mod 3 \\ &\implies 2\Big|(a-b) \text{ and } 3\Big|(a-b) \\ &\implies \text{(since gcd(2,3)=1)}, 6\Big|(a-b) \\ &\implies a = b \mod 6 \end{aligned}$$

Thus $f(a) = f(b) \implies a = b$ in $\mathbb{Z}_6$, so f is 1-1.

Solutions

Define $f : \mathbb{Z}_6 \to \mathbb{Z}_2 \oplus \mathbb{Z}_3$ by $f(m) = (m \mod 2, m \mod 3)$.

Is f onto?

For this function, we can see that it's onto by figuring out what f of each element is.

$$\begin{aligned} f(0) &= 0 \\ f(1) &= (1,1) \\ f(2) &= (0,2) \\ f(3) &= (1,0) \\ f(4) &= (0,1) \\ f(5) &= (1,2) \end{aligned}$$

Thus for every $(m,n) \in \mathbb{Z}_2 \oplus \mathbb{Z}_3$, there exists an $a \in \mathbb{Z}_6$ such that $f(a) = (m,n)$, so f is onto.

(This shows one-to-one as well, of course)

Alternative Approach to showing f is onto:

Define $f : \mathbb{Z}_6 \to \mathbb{Z}_2 \oplus \mathbb{Z}_3$ by $f(m) = (m \mod 2, m \mod 3)$.

2 and 3 relatively prime $\Longrightarrow \exists s, t$ such that $1 = 2s + 3t$.
In fact, $1 = 2(-1) + 3(1)$.

Let $(a, b) \in \mathbb{Z}_2 \oplus \mathbb{Z}_3$. Need to find a $g \in \mathbb{Z}_6$ such that $f(g) = (a, b)$.

Define $g = 3ta + 2sb$. That is, define $g = 3(1)a - 2(-1)b = 3a - 2b$.

NTS $f(g) = (a, b)$.

Then

$$\begin{aligned} f(g) &= f(3a-2b) \\ &= (3a-2b \mod 2, 3a-2b \mod 3) \\ &= (3a \mod 2, -2b \mod 3) \\ &= ((3 \mod 2)a, (-2 \mod 3)b) \\ &= (a, b). \end{aligned}$$

Solutions

Define $f : \mathbb{Z}_6 \to \mathbb{Z}_2 \oplus \mathbb{Z}_3$ by $f(m) = (m \mod 2, m \mod 3)$.

Is f operation preserving?

The operation in $\mathbb{Z}_6$ is addition mod 6. The operation in $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ is addition mod 2 in the first component, addition mod 3 in the second component. We'll denote this operation as $\star$, just to indicate where it's showing up.

$$\begin{aligned} f(a+b \mod 6) &= (a+b \mod 2, a+b \mod 3) \\ &= (a \mod 2, a \mod 3) \\ &\qquad \star(b \mod 2, b \mod 3) \\ &= f(a) \star f(b). \end{aligned}$$

Thus f is operation preserving.