Recall:

- When using cycle notation, we often leave out cycles of length 1. If a number does not appear in a permutation, we assume that it is mapped to itself.

  For example, we could write $\beta = \begin{pmatrix} 1 & 5 & 2 & 4 \end{pmatrix} \begin{pmatrix} 3 \end{pmatrix}$ as $\beta = \begin{pmatrix} 1 & 5 & 2 & 4 \end{pmatrix}$, omitting the $\begin{pmatrix} 3 \end{pmatrix}$.

- We need to be very careful: $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ could be an element of S_3 or S_4 or S_5,etc. In each case, it would represent very different functions.

  In S_3, it would represent the function $\alpha : \{1, 2, 3\} \to \{1, 2, 3\}$ by $\alpha(1) = 2$, $\alpha(2) = 3$, and $\alpha(3) = 1$. But in S_5, it would represent the function $\beta : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ by $\beta(1) = 2$, $\beta(2) = 3$, $\beta(3) = 1$, $\beta(4) = 4$, $\beta(5) = 5$.

Recall:

Friday, we found

- $\left|\begin{pmatrix}1 & 5 & 2 & 4\end{pmatrix}\right| = 4$
- $\left|\begin{pmatrix}1 & 4 & 2 & 3 & 5\end{pmatrix}\right| = 5$
- $\left|\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 5 & 4\end{pmatrix}\right| = 6$

Conjectures:

- Cycles of length n have order n.
- **Disjoint** products of cycles have order equal to either the multiple or the least common multiple of the lengths of the cycles.

Some Results About Permutations

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- **Theorem 5.2** If $\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_m \end{pmatrix}$ and $\beta = \begin{pmatrix} b_1 & b_2 & \dots & b_k \end{pmatrix}$ are disjoint, then $\alpha\beta = \beta\alpha$.

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- **Theorem 5.4** Every permutation in S_n for $n > 1$ can be written as a product of (not necessarily disjoint) transpositions.

Another Result About Permutations

- **Lemma:** If we decompose the identity permutation into a product of r 2-cycles, $\epsilon = \beta_1 \beta_2 \cdots \beta_r$ where the β_i's are 2-cycles, then r must be even. That is, the identity can not be written as the product of an *odd* number of 2-cycles.

In Class Work

1. For each of the following permutations, first write it as the product of (not necessarily disjoint) transpositions, and then decide whether it is even or odd.
   (a) $\begin{pmatrix} 1 & 5 & 2 & 4 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 6 & 2 \end{pmatrix}\begin{pmatrix} 1 & 4 & 3 & 6 \end{pmatrix}$

2. Let n be a positive integer. If n is odd, is an n-cycle an odd or an even permutation? How about if n is even?

3. Is the product of an even permutation and an odd permutation even or odd? How about the product of two odd permutations?

Solutions

1. For each of the following permutations, first write it as the product of (not necessarily disjoint) transpositions, and then decide whether it is even or odd.
   (a) $\begin{pmatrix} 1 & 5 & 2 & 4 \end{pmatrix}$

   $$\begin{pmatrix} 1 & 5 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 5 \end{pmatrix}\begin{pmatrix} 5 & 2 \end{pmatrix}\begin{pmatrix} 2 & 4 \end{pmatrix}$$

   Because my decomposition contained an odd number of (not disjoint) transpositions, $\begin{pmatrix} 1 & 5 & 2 & 4 \end{pmatrix}$ is an **odd** permutation.

   (b) $\begin{pmatrix} 1 & 6 & 2 \end{pmatrix}\begin{pmatrix} 1 & 4 & 3 & 6 \end{pmatrix}$

   $$\begin{aligned}
   \begin{pmatrix} 1 & 6 & 2 \end{pmatrix} &= \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 6 \end{pmatrix}\begin{pmatrix} 6 & 2 \end{pmatrix} \\
   \begin{pmatrix} 1 & 4 & 3 & 6 \end{pmatrix} &= \begin{pmatrix} 1 & 6 \end{pmatrix}\begin{pmatrix} 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 4 \end{pmatrix}\begin{pmatrix} 4 & 3 \end{pmatrix}\begin{pmatrix} 3 \end{pmatrix} \\
   \begin{pmatrix} 1 & 6 & 2 \end{pmatrix}\begin{pmatrix} 1 & 4 & 3 & 6 \end{pmatrix} &= \left[\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 6 \end{pmatrix}\right]\left[\begin{pmatrix} 1 & 6 \end{pmatrix}\begin{pmatrix} 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 4 \end{pmatrix}\right] \\
   &= \left[\begin{pmatrix} 1 & 6 \end{pmatrix}\begin{pmatrix} 6 & 2 \end{pmatrix}\right]\left[\begin{pmatrix} 1 & 4 \end{pmatrix}\begin{pmatrix} 4 & 3 \end{pmatrix}\begin{pmatrix} 3 & 6 \end{pmatrix}\right]
   \end{aligned}$$

   This permutation is again **odd**.

Solutions

2. Let n be a positive integer. If n is odd, is an n-cycle an odd or an even permutation? How about if n is even?

   For any n, we can write an n-cycle $\alpha = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix}$ as

   $$\alpha = \begin{pmatrix} a_1 & a_n \end{pmatrix} \begin{pmatrix} a_1 & a_{n-1} \end{pmatrix} \begin{pmatrix} a_1 & a_{n-2} \end{pmatrix} \cdots \begin{pmatrix} a_1 & a_3 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \end{pmatrix}.$$

   This is a product of $n-1$ transpositions.

   Thus if n is odd, α is even, and if n is even, α is odd.

Solutions

3. Is the product of an even permutation and an odd permutation even or odd? How about the product of two odd permutations?

   An even permutation will always decompose into the product of an even number of transpositions, while an odd permutation will always decompose into the product of an odd number of transpositions.

   Thus the product of an even and an odd permutation is the product of an even + an odd number of transpositions, which is always odd.

   Similarly, the product of two odd permutions is the product of an odd + an odd number of transpositions, which is always even.