Notation:

What we're trying to denote	If the operation is addition	anything but addition	
the inverse of g	$-g$	g^{-1}	
$g * g$ n times	ng	g^n	ng does **not** mean $n \times g$, it means $g + g + g \ldots + g$.
the identity	$0 = e$	$1 = e$	

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the identity	$0 = e$	$1 = e$	
g, 0 times	$0g = 0$	$g^0 = 1$	
$g^n * g^m$	$ng + mg$ $= (n+m)g$	$g^n g^m = g^{n+m}$	

Question:

Is $(gh)^n = g^n h^n$?

Definition:

If G is a group, then G is **Abelian** if $ab = ba$ for all $a, b \in G$. In other words, G is Abelian if and only if the operation is commutative.

In Class Work

Which of the following groups are **Abelian**?

1. $\mathbb{Z}_5$, under addition mod 5
2. $U(12)$, under multiplication mod 12
3. $\mathbb{Z}_2 \oplus \mathbb{Z}_3$, under $(+ \mod 12, + \mod 3)$
4. D_4, under composition
5. $S = \{f : \mathbb{R} \rightarrow \mathbb{R} | f \text{ is one-to-one and onto}\}$, under composition

In Class Work

1. What is **the order of** D_4? That is, what is $|D_4|$?

2. In D_4, what is **the order of** the element H (reflection across the horizontal axis)? That is, what is $|H|$? How about the order of the rotation R_{90}, $|R_{90}|$?

3. What is $|GL(2, \mathbb{R})|$?

4. In $\mathbb{Z}_8$, what is $|2|$? How about $|3|$?

Solutions:

1. What is *the order of* D_4? That is, what is $|D_4|$?

   Order of a *group*= the number of elements: $|D_4| = 8$.

2. In D_4, what is *the order of* the element the reflection H, $|H|$? How about the order of the rotation R_{90}, $|R_{90}|$?

   Order of an element = (in this case) smallest # of times we can do the motion to end up equivalent to the identity motion.

   $|H| = 2$, $|R_{90}| = 4$.

3. What is $|GL(2, \mathbb{R})|$?

   Since $GL(2, \mathbb{R})$ is the set of all 2×2 matrices with entries in $\mathbb{R}$ and non-zero determinants, $|GL(2, \mathbb{R})| = \infty$.

Solutions:

4. In $\mathbb{Z}_8$, what is $|2|$? How about $|3|$?

   $2+2 \bmod 8 = 4$, $2+2+2 \bmod 8 = 6$, $2+2+2+2 \bmod 8 = 0$. Thus the smallest number of 2's we can add to get the identity is 4, so $|2| = 4$.

   As for 3, we're not going to get to $3+3+\ldots+3 = 0 \bmod 8$ until we have eight 3's.

   (Check it out!)

   Thus $|3| = 8$.

1. Find $|\mathbb{Z}|$. In $\mathbb{Z}$, find $|2|$.
2. Find $|U(p)|$. In $U(5)$, find $|2|$.
   Remember, $U(p) = \{a \in \mathbb{Z}^+ | a < p \text{ and } gcd(a, p) = 1\}$ is a group under multiplication mod p. The identity is of course 1.
3. Find $|\mathbb{Z}_4 \oplus U(5)|$. In $\mathbb{Z}_4 \oplus U(5)$, find $|(2, 2)|$.
   Remember: $\mathbb{Z}_4 \oplus U(5) = \{(a, b) | a \in \mathbb{Z}_4, b \in U(5)\}$ is a group under the operation $(+ \mod 4, \cdot \mod 5)$. Also, because 0 is the identity of $\mathbb{Z}_4$ and 1 is the identity of $U(5)$, the identity element of $\mathbb{Z}_4 \oplus U(5)$ is $(0, 1)$.

1. $|\mathbb{Z}\,| = \infty$
   No matter how many times you add 2 to itself, you never get 0, so $|2| = \infty$.
   What's $|n|$, $n \neq 0$?
2. If p is prime, $|U(p)| = p - 1$, since every positive integer less than p is relatively prime to p.
   As for $|2|$ in $U(5)$:

$$\begin{aligned} 2 \cdot 2 \mod 5 &\equiv 4 \\ 2 \cdot 2 \cdot 2 \mod 5 &\equiv 3 \\ 2 \cdot 2 \cdot 2 \cdot 2 \mod 5 &\equiv 1 \end{aligned}$$

   Thus $|2|$ in $U(5)$ is 4.
3. $|\mathbb{Z}_4 \oplus U(5)|$:
   Since each element of $\mathbb{Z}_4$ can be paired with any element of $U(5)$, $|\mathbb{Z}_4 \oplus U(5)| = |\mathbb{Z}_4| \times |U(5)| = 4 \times 4 = 16$.
   As for $|(2,2)|$:

$$\begin{aligned} (2,2) * (2,2) &= (0,4) \\ (2,2) * (2,2) * (2,2) &= (2,3) \end{aligned}$$