(Several of the problems below are from *Lessons in Mathematics and Art*, Lessons 1 and 2).

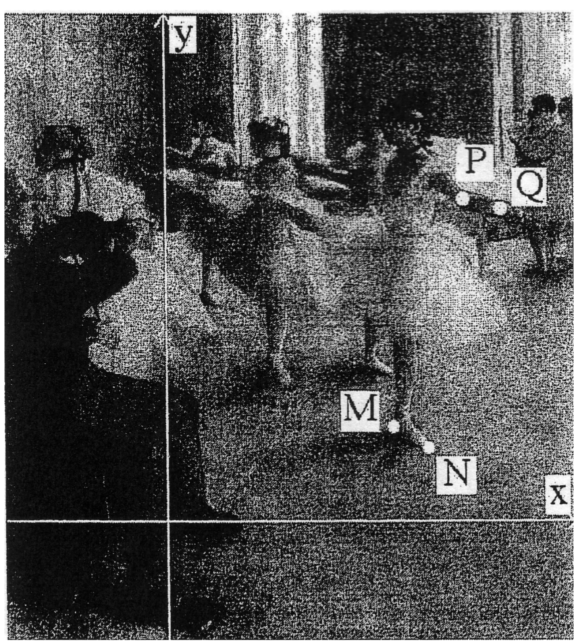
1. (This is an adaptation of Problem 2 from Lesson 1 of *Lessons in Mathematics and Art*; if you'd like to see the picture below in color, this text is available on the web at `http://mypage.iu.edu/~mathart/viewpoints/lessons/` .)

   Below is a detail from Edgar Degas' painting, *The Rehearsal*, with picture plane coordinate axes superimposed. Using the points $M(216, 88)$, $N(249, 68)$, $P(283, 302)$, and $Q(317, 293)$ in the figure (the coordinates are in pixels), find:

   (a) $d(M, N)$

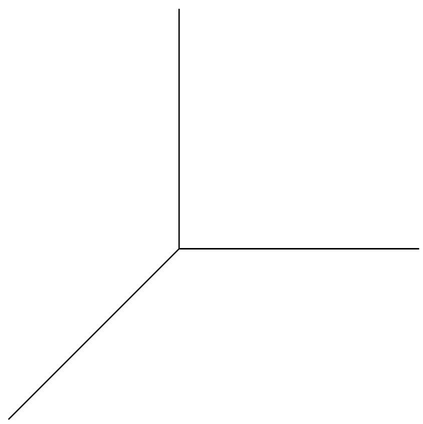
   (b) $d(P, Q)$

   (c) We've seen that in several systems of proportions, a person's foot should be about the same length as their forearm. In 3-space, the dancer's left foot and left forearm would be roughly parallel and directly above one another. We'll see later that because of that, if they were indeed the same length, then their images in the painting would also be the same length. Are they?

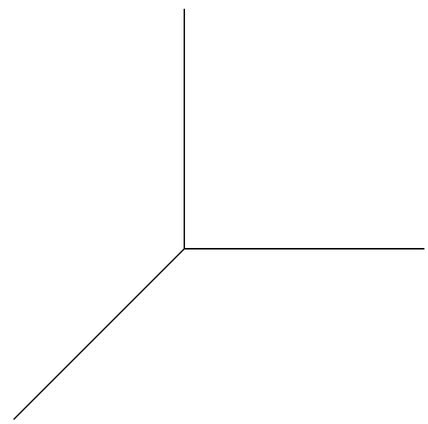


2. Please plot the following points on a set of 3-D coordinate axes.

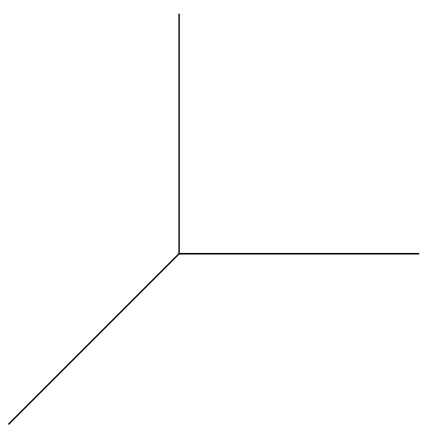
   (a) $A(1, 3, 4)$



   (b) $B(2, 4, 0)$



   (c) $C(0, 3, -1)$



3. In this problem, you're going to be considering a box whose faces are parallel to the coordinate planes. Suppose the corners of this box have the following coordinates:

| Bottom | Top |
|---|---|
| $A(1,3,4)$ | $E(1,7,4)$ |
| $B(8,3,4)$ | $F(8,7,4)$ |
| $C(8,3,10)$ | $G(8,7,10)$ |
| $D(1,3,10)$ | $H(1,7,10)$ |

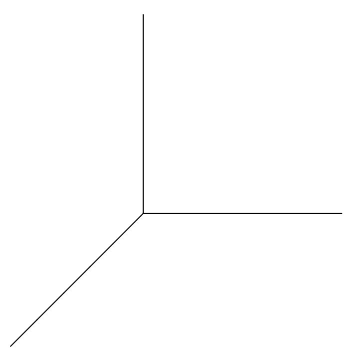
*Note:* You do not have to plot each of the 8 corners and draw the box in order to do this question, although you certainly can if you think that will help you think about the problem clearly.

(a) How deep is the box in the $x$ direction?

(b) How tall is the box in the $y$ direction?

(c) How wide is the box in the $z$ direction?

4. Now I want you to look for patterns in Problem **??**, and use them. We again have a box whose faces are parallel to the coordinate planes. Suppose the coordinates of two opposing corners of a box (the rear-left-bottom corner and the front-right-top corner) have coordinates $(3, -1, 5)$ and $(10, 7, 10)$.



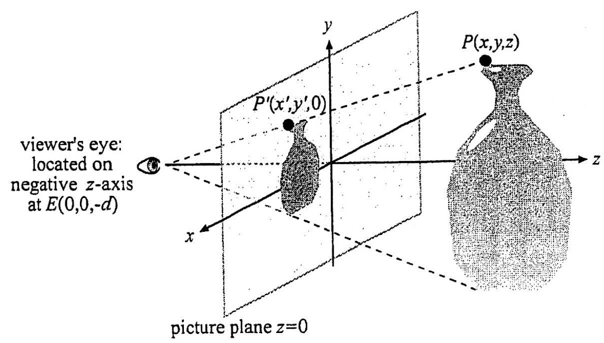
(a) How deep is the box in the $x$ direction?

(b) How tall is the box in the $y$ direction?

(c) How wide is the box in the $z$ direction?

(d) What are the coordinates of the remaining 6 corners?

5. In the figure below (which is also from Lesson 1 of *Lessons in Mathematics and Art*), suppose that $d = 3$ and suppose that the point $P(x, y, z)$ were moved so that $x = 0$, $y = 4$, and $z = 5$.




(a) Which coordinate plane would $P(x, y, z)$ lie in?

(b) Without using the Perspective Theorem, what would the image of $x$ in the picture plane, $x'$, be? (You can always check your answer using the formula, of course!)

(c) Again without using the Perspective Theorem, what would the image of $y$ in the picture plane, $y'$ be?

6. This exercise (from Lesson 2 of *Lessons in Mathematics and Art*) deals with a point $P$ whose $x$- and $y$- coordinates do not change (they are equal to 2 and 3, respectively), but whose $z$-coordinate gets bigger and bigger. That is, the point moves farther and farther away from the picture plane and the viewer.

Throughout this problem, use that the viewing distance $d$ is 5 units and use the Perspective Theorem.

(a) Suppose $P = (2, 3, 5)$. What are the values of $x'$ and $y'$?

(b) Now suppose $P = (2, 3, 95)$. What are $x'$ and $y'$?

(c) What if $P = (2, 3, 995)$?

(d) Draw one TOP VIEW and one SIDE VIEW like those we did in class (they're also in Figure 2 of Lesson 2 in *Lessons in Mathematics and Art*), and include all the points $P$ and $P'$ from parts (a)-(c), along with light rays to the viewer's eye (the drawings need not be to scale). Can you see what's happening?

(e) Consider a point $P(x, y, z)$. If $x$ and $y$ do not change, but $z$ gets bigger and bigger, what happens to the picture plane image $P'$ of $P$?

(f) Our everyday experience tells us that objects appear smaller as they get farther away. Explain how this is consistent with your answers to parts (a)-(e).

7. We are going to draw the same cube in different positions, using the Perspective Theorem. We'll use these pictures to begin to deduce for ourselves the rules of perspective. You *will* need graph paper! (I would suggest having each unit be several squares long, so your pictures are big enough to really appreciate.)

   In every case, our cube will have side of length 4, and the viewing distance $d$ (how far the viewer's eye is from the picture plane) will be 8.

   We will call our cube $ABCDEFGH$ with the base being square $ABCD$ and the top being square $EFGH$. (Note that $A$ is connected to $B$ and $D$, $B$ is connected to $A$ and $C$, etc; $E$ is directly above $A$, $F$ is directly above $B$, etc).

   (a) We'll begin with a cube whose top and bottom are horizontal and whose front and back are parallel to the picture plane. The bottom will be above the viewer's eye.

   Use the following coordinates for the corners of the cube:

| Base=ABCD | | Top=EFGH | |
|---|---|---|---|
| A | $(8, 3, 4)$ | E | $(8, 7, 4)$ |
| B | $(12, 3, 4)$ | F | $(12, 7, 4)$ |
| C | $(8, 3, 8)$ | G | $(8, 7, 8)$ |
| D | $(12, 3, 8)$ | H | $(12, 7, 8)$ |

   i. What are the coordinates for the viewer's eye?

   ii. Using the Perspective Theorem, find the coordinates for each of the 8 corners (shown again below) of the image in the picture plane (that is, find $(x', y')$). You may do these calculations by hand or, if you're comfortable with it, you may use a spreadsheet like Excel. If you use a spreadsheet, please include it with your work.

| Base=ABCD | | Top=EFGH | |
|---|---|---|---|
| A | $(8,3,4)$ | E | $(8,7,4)$ |
| B | $(12,3,4)$ | F | $(12,7,4)$ |
| C | $(8,3,8)$ | G | $(8,7,8)$ |
| D | $(12,3,8)$ | H | $(12,7,8)$ |

iii. *Carefully* plot the points you found in the part **??** in the $xy$ plane on graph paper. (Remember you are *not* using 3D space axes for this!) Then (paying attention to the right order), neatly connect them with straight lines (use a straight edge, and use dashed lines to indicate the hidden edges) to obtain the perspective image.

iv. Get a good idea of what the viewing distance is in the scale you used (that is, how long is 8 units?), and then put one eye at that distance from the page, directly opposite the origin. Look at your perspective image with that one eye. Do you see a cube, with the illusion of depth?

v. Your cube has one set of four parallel lines which are not parallel to the picture plane. Do those lines look parallel in the perspective image you've created? Using however many straight edges (pieces of paper, for instance) you need, see where they intersect (this may not be on your piece of graph paper). What can you say about where these four lines intersect?

(b) We'll continue with the same cube, but we'll turn it so that while the top and bottom are still horizontal, now one edge is facing us, rather than the front and back being parallel to the picture plane. We'll also move it so that the top is below the viewer's eye.

Use the following coordinates for the corners of the cube:

| Base=ABCD | | Top=EFGH | |
|---|---|---|---|
| A | $(0,-6,4)$ | E | $(0,-2,4)$ |
| B | $(2.8,-6,6.8)$ | F | $(2.8,-2,6.8)$ |
| C | $(0,-6,9.7)$ | G | $(0,-2,9.7)$ |
| D | $(-2.8,-6,6.8)$ | H | $(-2.8,-2,6.8)$ |

i. Find the coordinates for each of the corners of the image in the picture plane. Carefully plot them in the $xy$ plane on graph paper, then neatly connect them with straight lines to obtain the perspective image.

ii. As with the previous exercise, put one eye at the viewing distance opposite the origin. Look at your perspective image with that one eye. Does it leap off the page at you?

iii. Again, can you get a sense of where the parallel lines intersect?