Most of these exercises are either from Chapter 6 in Symmetry, Shape, and Space*, or are inspired by that chapter.*

1. In *Flatland*, A. Square has a conversation with his grandson, reproduced below:

   > *I began to show the boy how a Point by moving through a length of three inches makes a Line of three inches, which may be represented by 3; and how a Line of three inches, moving parallel to itself through a length of three inches, makes a Square of three inches every way, which may be represented by* 3^2 *[square inches].*

   Generalize to give a geometric significance to the quantity 3^3, thus answering a question posed by the grandson.

2. What would A. Square observe (assuming he has both the time and the effrontery to walk around and perhaps even touch), if a cube passed through Flatland:

   (a) so that one pair of faces is parallel to the plane of Flatland?

   (b) so that one corner approaching Flatland before the rest? (This one is tricky – do the best with it you can!)

3. What would a 4-dimensional cube look like to us if it passed through our 3-dimensional space "flush" with our space? (I'm of course having trouble finding words to describe, since we don't have many words to describe hyperspace, but what I'm asking you to do is the analogy of part (a) in the previous problem: one pair of cubes is "parallel" to our space). Describe and draw pictures as necessary to make your point.

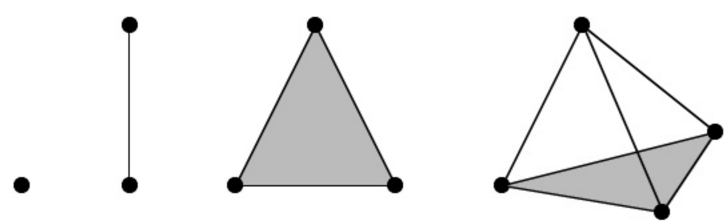
4. A circle is the set of all points in a plane equidistant from the center. A sphere is the set of all points in space equidistant from the center.

   (a) How would you define a fourth dimensional sphere, called a hypersphere?

   (b) What would we observe if a hypersphere passed through our space? Determine this by thinking analogously – what did A. Square observe as the sphere passed through his space?

5. Cubes and hypercubes are higher-dimensional analogues of the two-dimensional square. In Chapter 3 of *Dimensions*, (the chapter on the 4th dimension), you see the tetrahedron and the 4-dimensional analogue of the tetrahedron, both of which are higher-dimensional analogues of the two-dimensional triangle. On page 203-204 of *4th Dimension, Section 2* on OnCourse , your text discusses in detail how the three-dimensional tetrahedron is created, by moving from one dimension to the next as we did with the hypercube:

> *Start with a single point. In the next generation, add another point above the first, and connect the two, creating a line segment. Place that line segment on the floor and add another point above the line segment and connect this point with each of the points on the line segment, obtaining a triangle. Place the triangle flat on the floor and add another point above it; connecting the new point to each of the points in the triangle gives a three-dimensional solid called the tetrahedron or triangular pyramid.*



> *Adding another point in the fourth dimension,* ana *or* kata *the figure, and connecting this point to each point in the tetrahedron gives a figure called the* hypertetrahedron *or* pentahedroid.

Using this description, develop formulae for the hypertetrahedron analogous to those we developed for the hypercube, and use them to fill in the following table. The thought process will be essentially identical to that we used for the hypercube, but of course since the figures you're working with are different, the formulae you develop will also be different .

Dimension	0D	1D	2D	3D	4D
Figure	point	segment	triangle	tetrahedron	hypertetrahedron
vertices v					
edges e					
faces f					
solids s					
4D regions t					

6. If we can ponder a 4th spatial dimension, we can also ponder a 5th. In 5 dimensional space, there will be 5 different fundamental directions of movement: up/down, left/right, in/out, ana/kata plus the additional 5th dimensional directions – since ana=up and kata = down, perhaps we could use *upari* and *nana*, which are (according to the internet) Sanskrit for up and down.

   Define a **hyperhypercube** in the 5th dimension analogously to how we defined the hypercube in the 4th:

   > Begin with a hypercube. Place a copy of it parallel to the original in the fifth dimension in "just the right position" (as we do when forming squares and cubes). Attach each vertex of the original hypercube to the corresponding vertex of the

copy.

Throughout the following questions on the hyperhypercube, please feel free to refer to our work on the hypercube, either in your notes or on the relevant web page, or both. Each of the following questions is really two questions: a"how many" question, and a "why" question. Although you're welcome to use a formula to answer the "how many" portion of each question, when you answer the "why" portion, please discuss it in the context of how the hyperhypercube is formed, rather than simply referring to patterns in the numbers or to the existence of the formulas. In other words, what sort of thought processes did we use to develop the formulas?

(a) How many vertices does the hyperhypercube have, and why?

(b) How many edges does the hyperhypercube have, and why?

(c) How many 2-dimensional faces does the hyperhypercube have, and why?

(d) How many 3-dimensional solids does it have?

(e) How many 4-dimensional regions?

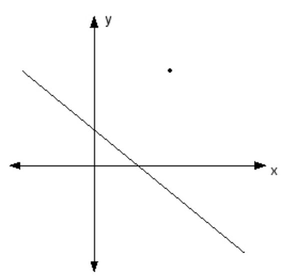
(f) How many 5-dimensional regions?

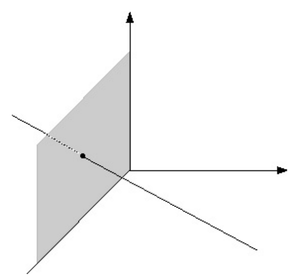
7. In the plane, note that any two non-parallel lines intersect in a point.

    (a) Generalize this idea to the intersection of two non-parallel planes in the third dimension. That is, what does the intersection of any two non-parallel planes look like? Use sheets of paper (or an equivalent) as planes to help you figure this out.

(b) Generalize even further, this time to the intersection of two non-parallel three-dimensional spaces in hyperspace. For this, of course, you'll need to use analogy to the dimensions you *can* picture. In other words, look for patterns in the two lower dimensions we've discussed so far.

8. In the illustration on the left, you see that "in general", (i.e. usually) a point and a line on the plane will not intersect. In other words, given a randomly chosen line and a randomly chosen point in the same plane, more likely than not the point will not be on the line. In the illustration on the right, you see that "in general", a line and a plane in 3 dimensional space will intersect in a point.





(Of course, this intersection is not always a point, but given an randomly chosen line in space and a randomly chosen plane in space, *more often than not* it is.)

(a) What is the most likely intersection of two lines in the plane? *Draw a sketch to illustrate.*

(b) What is the most likely intersection of two lines in 3-space? *Draw a sketch to illustrate.*

(c) Look at the results for this exercise, as well as those of part (a) of the previous exercise. In each case, we were looking at the intersection of either a point (0D), a line (1D), or plane (2D) with a line (1D) or plane (2D). Furthermore, in each case, the intersection occurred in either the plane (2D) or space (3D). And in each case, the resulting intersection was either nothing (no dimension), a point (0D), or a line (1D). In Part (b) of the previous exercise, you generalized the intersection of two 3D space in hyperspace.

Use your results to fill in the table on the next page that compare the dimension of the two intersecting objects, the dimension of the space the intersection occurs in, and the dimension of the most likely resulting intersection. (I've filled in the two facts that I gave you.)

Description of intersection	Dimension of object 1	Dimension of object 2	Dimension of space in which intersection occurs	Dimension of object formed by the intersection
Point with line in the plane	0	1	2	no intersection, so none (**not** 0)
Line with line in the plane				
Line with line in space				
Line with plane in space	1	2	3	point, so 0
Plane with plane in space				
Space with space in hyperspace (4D)				

(d) Scrutinize and ponder your completed table, looking for a relationship between the dimension of object 1, the dimension of object 2, the dimension of the space they sit in, and the dimension of the resulting intersection. Turn that relationship into a formula that gives the dimension of the resulting intersection if you know the other three quantitities: the dimensions of the two intersecting objects and the dimension of the space.

(e) What is the most likely intersection of a line and a plane in hyperspace? Justify your answer using whatever balance of insight into the 4th dimension and the formula you developed in the previous part that works for you.

(f) What is the most likely intersection of two planes in hyperspace? Again, justify your answer

(g) What is the most likely intersection of a plane and a 3-space in hyperspace?