Thinking About the 4th Dimension

- Ways we've seen to represent 3 dimensions in 2:
  - Perspective
  - Stereoscopic Projection (saw in *Dimensions*, Chapters 1 and 2)
  - Shadows (saw a bit in *Dimensions*, Chapter 3)
  - A series of cross-sections (talked about; saw in *Dimensions*, Chapter 2)
- Can we formulate ways to think of 4 dimensions in 3?

How Will We Approach the 4th Dimension?

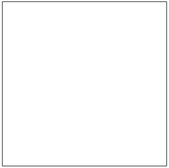
- Look for patterns in Dimensions 0, 1, 2, and 3
- Generalize what we can about 4D and higher, based on what we observe.
- In most cases, we will be **defining** what we mean by a specific 4 dimensional concept.
- (As you've seen in Chapter 3 of *Dimensions*) Most studies of the 4th dimension begin by looking at simple familiar geometric figures, extending them to their 4 (and higher) dimensional analogues.
- We will look at the 0, 1, 2, and 3 dimensional analogues of cubes and tetrahedrons:
  - How do we build the 1D version from the 0D version?
  - How do we build the 2D version from the 1D version?
  - How do we build the 3D version from the 2D version?
- In each case, we will observe that from one dimension to the next, some processes stay the same – so we will **define** the 4D analogue to be the geometric figure that we build by following that same procedure again.

Building A Cube:

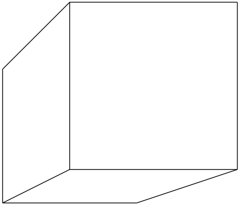
We use:

- points to create line segments

- line segments to create squares



- squares to create cubes



Recall:

In 3 dimensions, we have 3 sets of perpendicular directions:

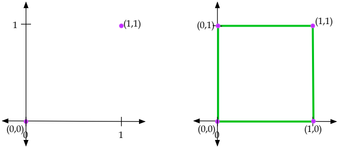
- Up/Down
- Left/Right
- In/Out

In 4 dimensions, we add a fourth set that is perpendicular to all three of these, called Ana and Kata.

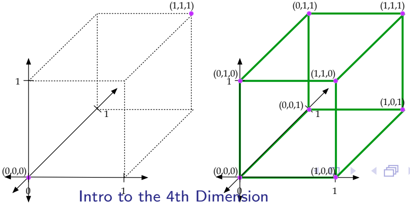
Thinking about the Hypercube:

- 1D: the unit line segment is defined by vertices 0 and 1.

- 2D: the unit square is defined by vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$.



- 3D: the unit cube is defined by vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, $(1,1,1)$.



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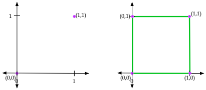

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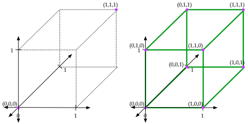


- 4D: Define: the hypercube is defined by vertices $(0,0,0,0)$, $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, $(1,1,0,0)$, $(1,0,1,0)$, $(1,0,0,1)$, $(0,1,1,0)$, $(0,1,0,1)$, $(0,0,1,1)$, $(1,1,1,0)$, $(1,1,0,1)$, $(1,0,1,1)$, $(0,1,1,1)$, $(1,1,1,1)$.

Thinking about the Hypercube:

Building one "cube" in the family from the previous dimension.

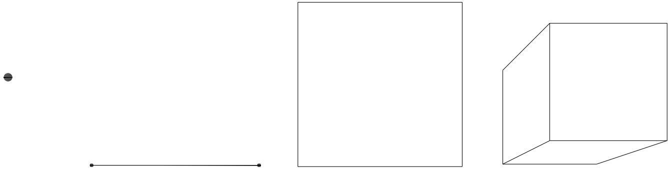
Recall: To go from:

- Point to Line Segment: Take one point, make a copy, place copy in parallel 0-dimensional space, attach.
- Line Segment to Square: Take one line segment, make a copy, place copy in parallel 1-dimensional space, line up corresponding vertices, and attach corresponding vertices.
- Square to Cube: Take one square, make a copy, place copy in parallel 2-dimensional space, line up corresponding vertices, and attach corresponding vertices.

Define: The hypercube is what we create when we take one cube, make a copy and place that copy in a parallel 3 dimensional space (an appropriate distance away), line up corresponding vertices and attach them.

Thinking About the Hypercube: Looking for Patterns in Lower Dimensions

0D	1D	2D	3D	4D
point	line segment	square	cube	hypercube

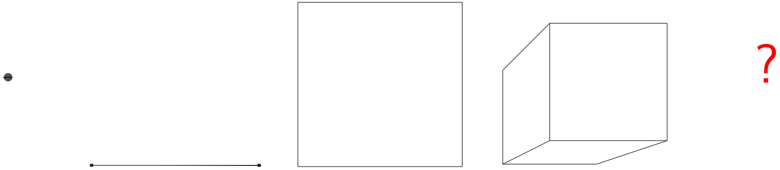


?

0D	1D	2D	3D	4D
1 vertex	2 vertices	4 vertices	8 vertices	?
0 edges	1 edge	4 edges	12 edges	?
0 faces	0 faces	1 face	6 faces	?
0 solids	0 solids	0 solids	1 solid	?

Thinking About the Hypercube: Looking for Patterns in Lower Dimensions

0D	1D	2D	3D	4D
point	line segment	square	cube	hypercube

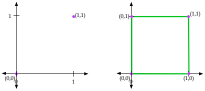


1 vertex	2 vertices	4 vertices	8 vertices	16?
0 edges	1 edge	4 edges	12 edges	?
0 faces	0 faces	1 face	6 faces	?
0 solids	0 solids	0 solids	1 solid	8?

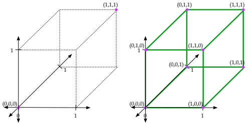
Thinking about the Hypercube:

- 1D: the unit line segment is defined by vertices 0 and 1.

- 2D: the unit square is defined by vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$.



- 3D: the unit cube is defined by vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, $(1,1,1)$.



- 4D: Define: the hypercube is defined by vertices $(0,0,0,0)$, $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, $(1,1,0,0)$, $(1,0,1,0)$, $(1,0,0,1)$, $(0,1,1,0)$, $(0,1,0,1)$, $(0,0,1,1)$, $(1,1,1,0)$, $(1,1,0,1)$, $(1,0,1,1)$, $(0,1,1,1)$, $(1,1,1,1)$.

Thinking about the Hypercube: How many vertices?

Dimension	# of Vertices
0D	$1 = 2^0$
1D	$2 = 2^1$
2D	$4 = 2^2$
3D	$8 = 2^3$

For all of our familiar dimensions ($n = 0$, 1, 2, or 3), the number of vertices a nth dimensional "cube" has is 2^n. **Why?**

Thinking about the Hypercube: How many vertices?

Dimension	# of Vertices
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For all of our familiar dimensions ($n = 0$, 1, 2, or 3), the number of vertices a nth dimensional "cube" has is 2^n. **Why?**

Forming a "n dimensional cube" from an $n-1$ dimensional cube

- Point to Line Segment: Take one point, make a copy, and attach.
- Line Segment to Square: Take one line segment, make a copy, and attach corresponding vertices.
- Square to Cube: Take one square, make a copy, and attach corresponding vertices.

Thinking about the Hypercube: How many vertices?

Dimension	# of Vertices
0D	$1 = 2^0$
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Forming a "n dimensional cube" from an $n-1$ dimensional cube

- Point to Line Segment: Take one point, make a copy, and attach. Attachment process creates no new vertices.
- Line Segment to Square: Take one line segment, make a copy, and attach corresponding vertices. Attachment process creates no new vertices.
- Square to Cube: Take one square, make a copy, and attach corresponding vertices. Attachment process adds no new vertices.

Thinking about the Hypercube: How many vertices?

Dimension	# of Vertices
0D	$1 = 2^0$
1D	$2 = 2^1$
2D	$4 = 2^2$
3D	$8 = 2^3$

For all of our familiar dimensions ($n = 0$, 1, 2, or 3), the number of vertices a nth dimensional "cube" has is 2^n. **Why?**

Forming a "n dimensional cube" from an $n - 1$ dimensional cube

- Point to Line Segment: Take one point, make a copy, and attach. Attachment process creates no new vertices. Line segment has twice the number of vertices as Point.
- Line Segment to Square: Take one line segment, make a copy, and attach corresponding vertices. Attachment process creates no new vertices. Square has twice the number of vertices as line segment.
- Square to Cube: Take one square, make a copy, and attach corresponding vertices. Attachment process adds no new vertices. Cube has twice the vertices as square.

Thinking about the Hypercube: How many vertices?

Dimension	# of Vertices
0D	$1 = 2^0$
1D	$2 = 2^1$
2D	$4 = 2^2$
3D	$8 = 2^3$

For all of our familiar dimensions ($n = 0$, 1, 2, or 3), the number of vertices a nth dimensional "cube" has is 2^n. **Why?**

Forming a "n dimensional cube" from an $n - 1$ dimensional cube

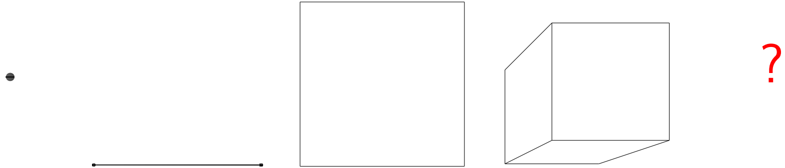
- Cube to Hypercube: By definition: we take one cube, make a copy, and attach corresponding vertices. Attachment process adds no new vertices. Hypercube has twice the vertices as a cube.

Thus for dimension n, where n=0, 1, 2, 3, or 4, it remains true that

$$\text{Number of Vertices of } n\text{-dimensional "cube"} = 2^n.$$

Thinking About the Hypercube:

0D	1D	2D	3D	4D
point	line segment	square	cube	hypercube



1 vertex	2 vertices	4 vertices	8 vertices	16 vertices
0 edges	1 edge	4 edges	12 edges	?
0 faces	0 faces	1 face	6 faces	?
0 solids	0 solids	0 solids	1 solid	?

Thinking about the Hypercube: How many edges?

Forming a "n dimensional cube" from an $n-1$ dimensional cube:

- Begin with an $n-1$-dimensional cube.
- Make a copy, place it in a parallel $n-1$ dimensional space. Distance away=length of cube's edge.
- Align copy so that perpendicular lines extending through each vertex of original cube go through the corresponding vertex of the copy.

How many edges?

Thinking about the Hypercube: How many edges?

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- Align copy so that perpendicular lines extending through each vertex of original cube go through the corresponding vertex of the copy.

How many edges?

In each case,

$$\# \text{ edges in dim } n = 2\big(\# \text{ edges in dim } n-1\big) + \# \text{ new edges}$$

Thinking about the Hypercube: How many edges?

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How many edges?

In each case,

$$\begin{aligned} \#\text{ edges in dim } n &= 2\big(\#\text{ edges in dim } n-1\big) + \#\text{ new edges} \\ &= 2\big(\#\text{ edges in dim } n-1\big) + \#\text{ vertices in dim } n-1 \end{aligned}$$

Thinking about the Hypercube: How many edges?

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How many edges?

In each case,

$$\begin{aligned}
\#\text{ edges in dim } n &= 2\big(\#\text{ edges in dim } n-1\big) + \#\text{ new edges} \\
&= 2\big(\#\text{ edges in dim } n-1\big) + \#\text{ vertices in dim } n-1 \\
E_n &= 2E_{n-1} + V_{n-1}
\end{aligned}$$

Thinking about the Hypercube: How many edges?

Let:

E_n = the number of edges of the n-dimensional cube

V_n = the number of vertices of the n-dimensional cube

Then

$$\boxed{E_n = 2E_{n-1} + V_{n-1}}$$

Check:

Dimension	# of Edges	# of Vertices
0	0	1
1	1	2
2	4	4
3	12	8

$$1 = 2 \times 0 + 1$$

$$4 = 2 \times 1 + 2$$

$$12 = 2 \times 4 + 4$$

Thinking about the Hypercube: How many edges?

We form a hypercube the same way:

- Begin with an 3-dimensional cube.
- Make a copy, place it in a parallel 3 dimensional space. Distance away=length of cube's edge.
- Align copy so that perpendicular lines extending through each vertex of original cube go through the corresponding vertex of the copy.

Thus we arrive at the number of edges the same way:

$$\boxed{E_4 = 2E_3 + V_3}$$

Dimension	# of Edges	# of Vertices
0	0	1
1	1	2
2	4	4
3	12	8

$$1 = 2 \times 0 + 1$$
$$4 = 2 \times 1 + 2$$
$$12 = 2 \times 4 + 4$$

Thinking about the Hypercube: How many edges?

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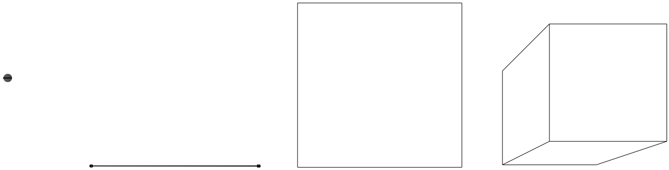
$$\boxed{E_4 = 2E_3 + V_3}$$

Dimension	# of Edges	# of Vertices
0	0	1
1	1	2
2	4	4
3	12	8

$$\begin{aligned} 1 &= 2 \times 0 + 1 \\ 4 &= 2 \times 1 + 2 \\ 12 &= 2 \times 4 + 4 \\ E_4 &= 2 \times 12 + 8 = 32 \end{aligned}$$

Thinking About the Hypercube:



0D	1D	2D	3D	4D
point	line segment	square	cube	hypercube
				?
1 vertex	2 vertices	4 vertices	8 vertices	16 vertices
0 edges	1 edge	4 edges	12 edges	32 edges
0 faces	0 faces	1 face	6 faces	?
0 solids	0 solids	0 solids	1 solid	?

Thinking about the Hypercube: How many faces?

Forming a "n dimensional cube" from an $n-1$ dimensional cube:

- Begin with an $n-1$-dimensional cube.
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How many faces?

Thinking about the Hypercube: How many faces?

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- Align copy so that perpendicular lines extending through each vertex of original cube go through the corresponding vertex of the copy.

How many faces?

In each case,

$$\begin{aligned} \#\text{ faces in dim } n &= 2\big(\#\text{ faces in dim } n-1\big) + \#\text{ new faces} \\ &= 2\big(\#\text{ faces in dim } n-1\big) + \#\text{ edges in dim } n-1 \\ F_n &= 2F_{n-1} + E_{n-1} \end{aligned}$$

Thinking about the Hypercube: How many faces?

Let:

F_n = the number of faces of the n-dimensional cube

E_n = the number of edges of the n-dimensional cube

Then

$$F_n = 2F_{n-1} + E_{n-1}$$

Check:

Dimension	# of Faces	# of Edges
0	0	0
1	0	1
2	1	4
3	6	12

$$0 = 2 \times 0 + 0$$
$$1 = 2 \times 0 + 1$$
$$6 = 2 \times 1 + 4$$

Thinking about the Hypercube: How many faces?

We form a hypercube the same way:

- Begin with an 3-dimensional cube.
- Make a copy, place it in a parallel 3 dimensional space. Distance away=length of cube's edge.
- Align copy so that perpendicular lines extending through each vertex of original cube go through the corresponding vertex of the copy.

Thus we arrive at the number of faces the same way:

$$\boxed{F_4 = 2F_3 + E_3}$$

Dimension	# of Faces	# of Edges
0	0	0
1	0	1
2	1	4
3	6	12

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$$1 = 2 \times 0 + 1$$
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Thinking about the Hypercube: How many faces?

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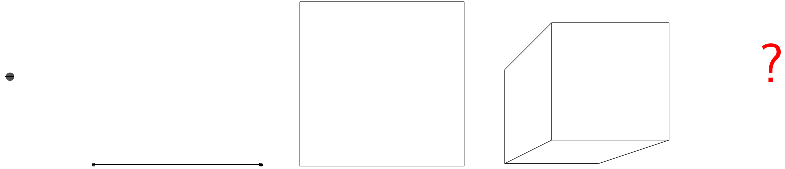
$$\boxed{F_4 = 2F_3 + E_3}$$

Dimension	# of Faces	# of Edges
0	0	0
1	0	1
2	1	4
3	6	12

$$\begin{aligned} 0 &= 2 \times 0 + 0 \\ 1 &= 2 \times 0 + 1 \\ 6 &= 2 \times 1 + 4 \\ F_4 &= 2 \times 6 + 12 = 24 \end{aligned}$$

Thinking About the Hypercube:

0D	1D	2D	3D	4D
point	line segment	square	cube	hypercube



1 vertex	2 vertices	4 vertices	8 vertices	16 vertices
0 edges	1 edge	4 edges	12 edges	32 edges
0 faces	0 faces	1 face	6 faces	24 faces
0 solids	0 solids	0 solids	1 solid	?

Thinking about the Hypercube: How many solids?

Forming a "n dimensional cube" from an $n-1$ dimensional cube:

- Begin with an $n-1$-dimensional cube.
- Make a copy, place it in a parallel $n-1$ dimensional space. Distance away=length of cube's edge.
- Align copy so that perpendicular lines extending through each vertex of original cube go through the corresponding vertex of the copy.

How many solids?

Thinking about the Hypercube: How many solids?

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- Make a copy, place it in a parallel $n-1$ dimensional space. Distance away=length of cube's edge.
- Align copy so that perpendicular lines extending through each vertex of original cube go through the corresponding vertex of the copy.

How many solids?

In each case,

$$\begin{aligned} \#\text{ solids in dim } n &= 2\big(\#\text{ solids in dim } n-1\big) + \#\text{ new solids} \\ &= 2\big(\#\text{ solids in dim } n-1\big) + \#\text{ faces in dim } n-1 \\ S_n &= 2S_{n-1} + F_{n-1} \end{aligned}$$

Thinking about the Hypercube: How many solids?

Let:

S_n = the number of solids of the n-dimensional cube

F_n = the number of faces of the n-dimensional cube

Then

$$S_n = 2S_{n-1} + F_{n-1}$$

Check:

Dimension	# of Solids	# of Faces
0	0	0
1	0	0
2	0	1
3	1	6

$$0 = 2 \times 0 + 0$$

$$0 = 2 \times 0 + 0$$

$$1 = 2 \times 0 + 1$$

Thinking about the Hypercube: How many solids?

We form a hypercube the same way:

- Begin with an 3-dimensional cube.
- Make a copy, place it in a parallel 3 dimensional space. Distance away=length of cube's edge.
- Align copy so that perpendicular lines extending through each vertex of original cube go through the corresponding vertex of the copy.

Thus we arrive at the number of faces the same way:

$$\boxed{S_4 = 2S_3 + F_3}$$

Dimension	# of Solids	# of Faces
0	0	0
1	0	0
2	0	1
3	1	6

$$0 = 2 \times 0 + 0$$
$$0 = 2 \times 0 + 0$$
$$1 = 2 \times 0 + 1$$

Thinking about the Hypercube: How many solids?

We form a hypercube the same way:

- Begin with an 3-dimensional cube.
- Make a copy, place it in a parallel 3 dimensional space. Distance away=length of cube's edge.
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Thus we arrive at the number of faces the same way:

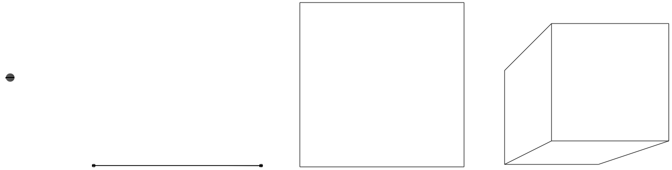
$$\boxed{S_4 = 2S_3 + F_3}$$

Dimension	# of Solids	# of Faces
0	0	0
1	0	0
2	0	1
3	1	6

$$\begin{aligned} 0 &= 2\times 0 + 0 \\ 0 &= 2\times 0 + 0 \\ 1 &= 2\times 0 + 1 \\ S_4 &= 2\times 1 + 6 = 8 \end{aligned}$$

Thinking About the Hypercube:

0D	1D	2D	3D	4D
point	line segment	square	cube	hypercube



?

0D	1D	2D	3D	4D
1 vertex	2 vertices	4 vertices	8 vertices	16 vertices
0 edges	1 edge	4 edges	12 edges	32 edges
0 faces	0 faces	1 face	6 faces	24 faces
0 solids	0 solids	0 solids	1 solid	8 solids
0 hyper solids	0 hyper solids	0 hyper solids	0 hyper solids	1 hyper solids

Explaining a Cube to A Square - Showing Him an Unfolded Cube

Would this image help A. Square understand what a cube is?