How Close Is Close Enough?

Goal: Develop standards to use when investigating whether an artist or architect may have intended the ratio of two dimensions to be some specific number.

- Actual lengths are unknowable
- The measured lengths are not the actual lengths. Use margins of error, or an error range, about each measurement in which we are confident the actual length must fall.
- **Goal**: If we have two measured lengths, A_m and B_m, and the ratio of the measured lengths $\frac{A_m}{B_m}$ is close to an interesting ratio R, develop a standard for judging if $\frac{A_m}{B_m}$ is close enough to R to conclude the closeness may have been intentional.

Recall:

- Always give a margin of error for your measurements.
  - Must be big enough so sure actual length is really in that range
  - At the same time, should be as small as possible (and still be true)
- If you find a length to be 20 cm ± 1 cm , then

$$\begin{aligned} \text{Measured length } L_m\text{: } L_m &= 20 \text{ cm} \\ \text{Actual length } L_a\text{: } 19 \text{ cm} \leq L_a &\leq 21 \text{ cm} \end{aligned}$$

- If you instead find the length to be 20 cm $\pm 1\%$, then

$$\begin{aligned} \text{Measured length } L_m\text{: } L_m &= 20 \text{ cm} \\ \text{Actual length } L_a\text{: } 0.99(20) \text{ cm} \leq L_a &\leq 1.01(20) \text{ cm} \\ \Rightarrow 19.8 \text{ cm} \leq L_a &\leq 20.2 \text{ cm}. \end{aligned}$$

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Approach: Use margins of error for the measurements A_m and B_m to find a range surrounding $\frac{A_m}{B_m}$, so that if R lies in that range, we'll say the **actual ratio** $\frac{A_a}{B_a}$ could be R.

Example:

- Measured ratio of width to height in a painting: $\frac{W_m}{H_m} = 1.4$
- We wonder: did the artist intend actual ratio $\frac{W_a}{H_a}$ to be $\sqrt{2} \approx 1.414$?
- Use knowledge of accuracy of measurements (i.e. known margins of error) for W_m and H_m to find a range about 1.4 in which I am sure the *actual ratio* $\frac{W_a}{H_a}$ falls.
- Suppose we find that range is $1.39 \leq \frac{W_a}{H_a} \leq 1.41$.
- 1.414 doesn't fall in that range.
  Reject that *actual ratio* $\frac{W_a}{H_a} = \sqrt{2}$.

Finding range of values for a ratio, 1% error :

Suppose we know:

- The measured height is within 1% of the actual height:

$$.99h_m \leq h_a \leq 1.01h_m$$

- The measured width is within 1% of the actual width:

$$.99w_m \leq w_a \leq 1.01w_m.$$

Then the actual ratio of these lengths also falls in a range:

$$\frac{\text{smallest } h_m}{\text{largest } w_m} \leq \frac{h_a}{w_a} \leq \frac{\text{largest } h_m}{\text{smallest } w_m}$$

$$\frac{.99h_m}{1.01w_m} \leq \frac{h_a}{w_a} \leq \frac{1.01h_m}{.99w_m}$$

$$.98\left(\frac{h_m}{w_m}\right) \leq \frac{h_a}{w_a} \leq 1.02\left(\frac{h_m}{w_m}\right)$$

Conclusion: if margin of error for both measurements is $\pm 1\%$, then the margin of error for the ratio will be (more or less) $\pm 2\%$.

Finding range of values for a ratio, 2% error :

Suppose we know:

- The measured height is within 2% of the actual height:

$$.98h_m \leq h_a \leq 1.02h_m$$

- The measured width is within 2% of the actual width:

$$.98w_m \leq w_a \leq 1.02w_m.$$

Then the actual ratio of these lengths also falls in a range:

$$\frac{\text{smallest } h_m}{\text{largest } w_m} \leq \frac{h_a}{w_a} \leq \frac{\text{largest } h_m}{\text{smallest } w_m}$$

$$\frac{.98h_m}{1.02w_m} \leq \frac{h_a}{w_a} \leq \frac{1.02h_m}{.98w_m}$$

$$.96\left(\frac{h_m}{w_m}\right) \leq \frac{h_a}{w_a} \leq 1.04\left(\frac{h_m}{w_m}\right)$$

Conclusion: if margin of error for both measurements is $\pm 2\%$, then the margin of error for the ratio will be (more or less) $\pm 4\%$.