Integration by Substitution:

- Identify u (usually the inside of a composition); use that to find $du = u'\ dx$
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- Replace the appropriate expressions of x in the integrand with u and du. du can not be in the denominator, raised to any power, or otherwise be inside any function.
- Your substitution has been successful if (1) there are no more x's or dx – and (2) if your new integral is antidifferentiable.
- Integrate the new simpler function.
- For an indefinite integral: back substitute: Replace u with its original more complicated formulation in terms of x.

Integration by Substitution:

- Identify u (usually the inside of a composition); use that to find $du = u' \, dx$
- For a definite integral: the limits depend on the variable. Either remember to back-substitute at the end, or change the limits now: when $x = a$, what is $u(a)$? when $x = b$, what is $u(b)$?
- Replace the appropriate expressions of x in the integrand with u and du. du can not be in the denominator, raised to any power, or otherwise be inside any function.
- Your substitution has been successful if (1) there are no more x's or dx – and (2) if your new integral is antidifferentiable.
- Integrate the new simpler function.
- For a definite integral: If you didn't change your limits of integration, back-substitute and then evaluate using $x = b$ and $x = a$. If you did change the limits of integration, do **not** back-substitute; evaluate using $u = u(b)$ and at $u = u(a)$.

In Class Work

Evaluate the following definite integrals:

1. $\int_0^{\pi/4} \sin(3\pi + 2x)\ dx$
2. $\int_0^4 (6 - 4x)^4\ dx$
3. $\int_0^1 \frac{2x^3}{1 + x^4}\ dx$
4. $\int_1^e \frac{\ln x}{x}\ dx$
5. $\int_1^4 \frac{\sqrt{1 + \frac{1}{x}}}{x^2}\ dx$
6. $\int_0^1 x e^{x^2}\ dx$
7. $\int_1^2 x(3x + 1)^5\ dx$
8. $\int_0^1 x\sqrt{3 - 2x}\ dx$
9. $\int_2^5 \frac{x + 1}{\sqrt[3]{3x^2 + 6x + 5}}\ \mathrm{dx}$
10. $\int_0^1 \frac{x^3}{1 + x^2}\ dx$
11. $\int_{-1}^0 \frac{e^{3x}}{(1 + e^{3x})^5}\ dx$
12. $\int_0^{\pi/4} x \tan(x^2)\ dx$

Solutions

1. $\displaystyle\int_0^{\pi/4} \sin(3\pi + 2x)\, dx$

- Choose $u = 3\pi + 2x$. Then $du = 2\, dx$, so $\frac{1}{2}\, du = dx$.
- Adjust limits: When $x = 0$, $u = 3\pi$; when $x = \frac{\pi}{4}$, $u = \frac{7\pi}{2}$.
- Substitute: $\displaystyle\int_0^{\pi/4} \sin(3\pi + 2x)\, dx = \frac{1}{2}\int_{3\pi}^{7\pi/2} \sin(u)\, du$
- Integrate:

$$\int_0^{\pi/4} \sin(3\pi + 2x)\, dx = \frac{1}{2}\int_{3\pi}^{7\pi/2} \sin(u)\, du = -\frac{1}{2}\cos(u)\Big|_{3\pi}^{7\pi/2}$$

- Evaluate:

$$\begin{aligned}\int_0^{\pi/4} \sin(3\pi + 2x)\, dx &= -\frac{1}{2}\cos(u)\Big|_{3\pi}^{7\pi/2} = -\frac{1}{2}\big(\cos(7\pi/2) - \cos(3\pi)\big) \\ &= -\frac{1}{2}\big(0 - (-1)\big) = -\frac{1}{2}\end{aligned}$$

Solutions

2. $\int_0^4 (6-4x)^4 \, dx$

- Choose $u = 6-4x$. Then $du = -4\, dx$, so $-\frac{1}{4}\, du = dx$.
- Adjust limits: when $x=0$, $u=6$; when $x=4$, $u=-10$.
- Substitute: $\int_0^4 (6-4x)^4 \, dx = -\frac{1}{4}\int_6^{-10} u^4 \, du$
- Integrate:

$$\int_0^4 (6-4x)^4 \, dx = -\frac{1}{4}\int_6^{-10} u^4 \, du = -\frac{1}{4}\cdot\frac{1}{5}u^5\Big|_6^{-10}$$

- Evaluate:

$$\begin{aligned}\int_0^4 (6-4x)^4 \, dx &= -\frac{1}{4}\frac{1}{5}u^5\Big|_6^{-10} = -\frac{1}{20}\left((-10)^5 - 6^5\right)\\ &= -\frac{1}{20}(-100000-7776) = \frac{107776}{20}\\ &= 5388.8\end{aligned}$$

Solutions

3. $\displaystyle\int_0^1 \frac{2x^3}{1+x^4}\,dx$

- Choose $u = 1 + x^4$.
  Then $du = 4x^3\,dx \Rightarrow \frac{1}{2}\,du = 2x^3\,dx$
- Adjust limits: when $x = 0$, $u = 1$; when $x = 1$, $u = 2$.
- Substitute:
$$\int_0^1 \frac{2x^3}{1+x^4}\,dx = \frac{1}{2}\int_1^2 \frac{1}{u}\,du$$
- Integrate:
$$\int_0^1 \frac{2x^3}{1+x^4}\,dx = \frac{1}{2}\int_1^2 \frac{1}{u}\,du = \ln|u|\Big|_1^2$$
- Evaluate:
$$\int_0^1 \frac{2x^3}{1+x^4}\,dx = \frac{1}{2}\ln|u|\Big|_1^2 = \frac{1}{2}\big(\ln(2) - \ln(1)\big) = \frac{\ln(2)}{2}$$

Solutions

4. $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$

- Choose $u = \ln(x)$. Then $du = \frac{1}{x}\,dx$.
- Adjust limits: when $x = 1$, $u = 0$; when $x = e$, $u = \ln(e) = 1$.
- Substitute:

$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du$$

- Integrate:

$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \frac{1}{2}u^2\Big|_0^1$$

- Evaluate:

$$\int_1^e \frac{\ln x}{x}\,dx = \frac{1}{2}u^2\Big|_0^1 = \frac{1}{2}(1-0)$$

Solutions

5. $\displaystyle\int_1^4 \frac{\sqrt{1+\frac{1}{x}}}{x^2}\,dx$

- Choose $u = 1 + \frac{1}{x}$. Then $du = -\frac{1}{x^2}\,dx \Rightarrow -du = \frac{1}{x^2}\,dx$.
- Adjust limits: when $x = 1$, $u = 2$; when $x = 4$, $u = \frac{5}{4}$.
- Substitute:

$$\int_1^4 \frac{\sqrt{1+\frac{1}{x}}}{x^2}\,dx = -\int_2^{5/4} \sqrt{u}\,du$$

- Integrate:

$$\int_1^4 \frac{\sqrt{1+\frac{1}{x}}}{x^2}\,dx = -\int_2^{5/4} \sqrt{u}\,du = -\frac{2}{3}u^{3/2}\Big|_2^{5/4}$$

- Evaluate:

$$\int_1^4 \frac{\sqrt{1+\frac{1}{x}}}{x^2}\,dx = -\int_2^{5/4} \sqrt{u}\,du = -\frac{2}{3}u^{3/2}\Big|_2^{5/4} = -\frac{2}{3}\left((5/4)^{3/2} - (2)^{5/4}\right)$$

Solutions

6. $\int_0^1 xe^{x^2}\,dx$

- Choose $u = x^2$. Then $du = 2x\,dx \Rightarrow \frac{1}{2}\,du = x\,dx$
- Adjust limits: when $x = 0$, $u = 0$, when $x = 1$, $u = 1$.
- Substitute:
$$\int_0^1 xe^{x^2}\,dx = \frac{1}{2}\int_0^1 e^u\,du$$
- Integrate:
$$\int_0^1 xe^{x^2}\,dx = \frac{1}{2}\int_0^1 e^u\,du = e^u\Big|_0^1$$
- Evaluate:
$$\int_0^1 xe^{x^2}\,dx = \frac{1}{2}\int_0^1 e^u\,du = e^u\Big|_0^1 = e - 1$$