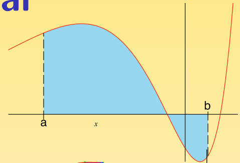
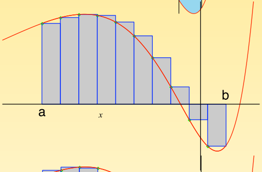
Definitions of the Definite Integral

$\displaystyle\int_a^b f(x)\,dx \overset{def}{=}$ signed area btw $f(x)$ & the x-axis from $x = a$ to $x = b$



$\displaystyle\int_a^b f(x)\,dx =$the limit as number of subintervals approaches ∞ of the left-hand sums



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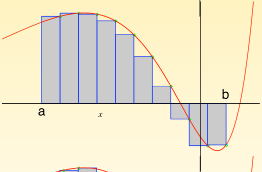
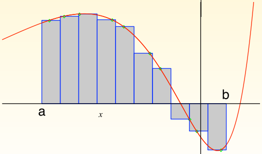


$\displaystyle\int_a^b f(x)\,dx =$the limit as number of subintervals approaches ∞ of any Riemann Sum



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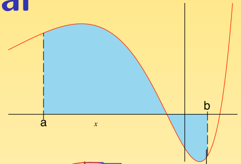
$\int_a^b f(x)\,dx \overset{def}{=}$ signed area btw $f(x)$ & the x-axis from $x = a$ to $x = b$



$$\int_a^b f(x)\,dx = \sum_{k=0}^{n-1} f(k\Delta x)\Delta x$$

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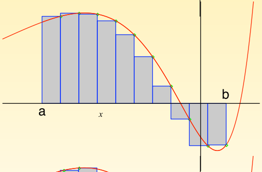
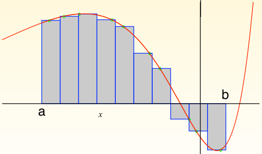


$$\int_a^b f(x)\,dx = \sum_{k=1}^{n} f(c_k)\Delta x$$

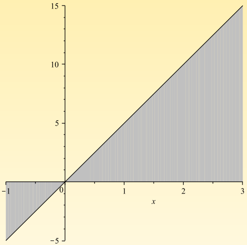


Example:

Consider $\int_{-1}^{3} 5x\ dx$.

1. Is $\int_{-1}^{3} 5x\ dx$ positive or negative?

signed area from $x = -1$ to $x = 3$.



More area is above the x-axis than below $\Rightarrow$ signed area is positive, so $\int_{-1}^{3} 5x\ dx > 0$.

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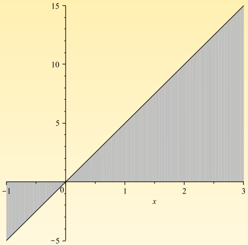


More area is above the x-axis than below $\Rightarrow$ signed area is positive, so $\int_{-1}^{3} 5x\,dx > 0$.

2. What *is* $\int_{-1}^{3} 5x\,dx$?

$$\begin{aligned}\int_{-1}^{3} 5x\,dx &= \text{Signed area of } \Delta \text{ from } -1 \text{ to } 0 \\ &\quad \text{(negative)} \\ &\quad + \text{Signed area of } \Delta \text{ from } 0 \text{ to } 3 \\ &\quad \text{(positive)} \\ &= -\frac{1}{2}(1)(5) + \frac{1}{2}(3)(15) \\ &= -\frac{5}{2} + \frac{45}{2} = 20\end{aligned}$$

In Class Work

Using only what we know so far – that the integral is the *signed* area between the graph and the x-axis – evaluate the following integrals. *No shortcuts!*

1. $\int_{-1}^{1} x^3 \, dx$

2. $\int_{0}^{4} 2x \, dx$

3. $\int_{-1}^{0} 2x \, dx$

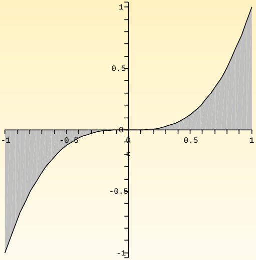
4. $\int_{-1}^{4} 2x \, dx$

5. $\int_{4}^{0} 2x \, dx$

Solutions

1. $\int_{-1}^{1} x^3 \, dx$

Sketch region between $y = x^3$ and x-axis from $x = -1$ to $x = 1$:

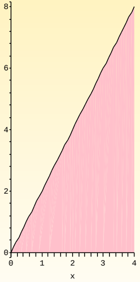


- Half of this graph is above the x-axis while the other half is below.
- The two sides are symmetric across the origin – so the negative part will cancel out the positive part.
- $\int_{-1}^{1} x^3 \, dx = 0$

Solutions:

2. $\int_0^4 2x\,dx$

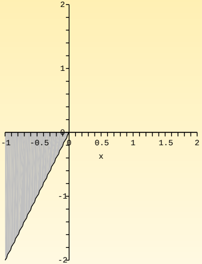
Sketch region between $y = 2x$ and x-axis from $x = 0$ to $x = 4$:



- All of the area is above the x-axis, so the integral will be positive.
- This is just a triangle with base 4 and height 8.
- $\int_0^4 2x\,dx = \frac{1}{2}(4)(8) = 16.$

Solutions:

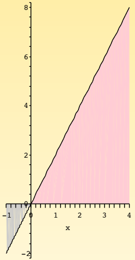
3. $\int_{-1}^{0} 2x\; dx$



- All of the area is below the x-axis, so the integral will be negative.
- This is just a triangle with base 1 and height 2.
- $\displaystyle\int_{-1}^{0} 2x\; dx = -\frac{1}{2}(1)(2) = -1.$

Solutions:

4. $\int_{-1}^{4} 2x \, dx$



- The region from -1 to 0 is negative, and the region from 0 to 4 is positive.
- $\int_{-1}^{4} 2x \, dx = \int_{-1}^{0} 2x \, dx + \int_{0}^{4} 2x \, dx = -1 + 16 = 15.$

$$\int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx$$

Solutions:

5. $\int_4^0 2x\ dx$

We found in #2 that $\int_0^4 2x\ dx = 16$.

But what does it mean to go backwards, from $x = 4$ to $x = 0$?

Solutions:

5. $\int_4^0 2x\, dx$

We found in #2 that $\int_0^4 2x\, dx = 16$.

But what does it mean to go backwards, from $x = 4$ to $x = 0$?

If we were just dealing with regular area, it doesn't mean anything.

Since the definite integral is signed area from a to b, direction matters.

The signed area moving from right to left should be different from moving from left. Since the absolute value of it should be the traditional area, it can only differ by the sign.

$$\int_4^0 2x\, dx = -\int_0^4 2x\, dx = -16.$$

$$\boxed{\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx}$$