Recall:

- **Thm 4.5.2: The Fundamental Theorem of Calculus, 1st form**
  Let f be continuous on an open interval I containing a. The function A_f defined by
  $$A_f(x) = \int_a^x f(t)\,dt$$
  is defined for all $x \in I$ and $\frac{d}{dx}(A_f(x)) = f(x)$. That is, A_f is an *antiderivative* of f. In other words,
  $$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$
- **Consequence:** If f is continuous, then f has an antiderivative– $A_f(x)$.
- **Thm 4.5.1: The Fundamental Theorem of Calculus, 2nd form**
  Let f be continuous on $[a, b]$, and let F be **any** antiderivative of f. Then
  $$\int_a^b f(x)\,dx = F(b) - F(a)$$

In Class Work

1. Evaluate the following definite integrals (begin with (a)-(c), move onto #2; only do (d)-(f) if you have extra time)

   (a) $\int_{-1}^{2} e^x \, dx$

   (b) $\int_{1}^{4} x^3 - 2x \, dx$

   (c) $\int_{1}^{3} 3x^2 \ln(x) + x^3 \left(\frac{1}{x}\right) \, dx$

   (d) $\int_{0}^{1} x^{12} e^{x^{13}} \, dx$

   (e) $\int_{0}^{3} 4e^x x + 2e^x x^2 \, dx$

   (f) $\int_{-1}^{1} \sqrt{1 - x^2} \, dx$

   Hint: Draw a picture of $\sqrt{1 - x^2}$

2. Let $f(t) = 2t\cos(t^2)$ and $F(x) = \int_{1}^{x} f(t) \, dt$.

   (a) Find the equation of the line tangent to $y = F(x)$ at $x = 3$.

   (b) Find a formula for $\frac{d}{dx}\left(F(x^3)\right)$.

Solutions:

1(a) $\int_{-1}^{2} e^x \, dx$

e^x is of course an antiderivative of e^x, so from the FTC v1,

$$\int_{-1}^{2} e^x \, dx = e^x \Big|_{-1}^{2} = e^2 - e^{-1}$$

1(b) $\int_{1}^{4} x^3 - 2x \, dx$

$\frac{x^4}{4} - x^2$ is an antiderivative of $x^3 - 2x$, so from the FTC v1,

$$\int_{1}^{4} x^3 - 2x \, dx = \left(\frac{x^4}{4} - x^2\right)\Big|_{1}^{4} = \left(\frac{4^4}{4} - 16\right) - \left(\frac{1}{4} - 1\right) = 48 + \frac{3}{4}$$

Solutions

1(c) $\int_1^3 3x^2 \ln(x) + x^3 \left(\frac{1}{x}\right)\, dx$

- **Goal 1:** find an antiderivative $F(x)$ of $3x^2 \ln(x) + x^3 \left(\frac{1}{x}\right)$.
  - The term on the right, $x^3 \left(\frac{1}{x}\right)$, can be simplified to x^2, which we know how to antidifferentiate. If the other term, $3x^2 \ln(x)$ were easy to antidifferentiate, we would just antidifferentiate each piece separately.
  - However, we have no idea what an antiderivative of $3x^2 \ln(x)$ is.
  - Both terms are products.
  - The two ways we're most familiar with that a function can differentiate into a product is if the original function – the one we're trying to find, the one that differentiates to what we have – is a composition or a product.
  - When you differentiate a composition, the result is also a composition.
  - There is no composition in $3x^2 \ln(x)$. Therefore this product most likely did not come from differentiating a composition.
  - When we differentiate a product, we would get a sum of two products, which we have.

Solutions

1(c) $\int_1^3 3x^2 \ln(x) + x^3 \left(\frac{1}{x}\right)\ dx$

- **Goal 1:** find an antiderivative $F(x)$ of $3x^2 \ln(x) + x^3 \left(\frac{1}{x}\right)$.
  - The product rule says that if $F = fg$, then $F' = fg' + f'g$ – in other words, the two products that make up the sum are very closely related – each one has an undifferentiated function (either f or g) and a differentiated function (either g' or f').
  - When I look more closely at $3x^2 \ln(x) + x^3 \left(\frac{1}{x}\right)$, I see that $3x^2$ is the derivative of x^3 and $\dfrac{1}{x}$ is the derivative of $\ln(x)$. That tells me that f and g are x^3 and $\ln(x)$.
  - Try: $F(x) = x^3 \ln(x)$.
  - *Check:* $F'(x) = (x^3)' \cdot \ln(x) + x^3 \cdot (\ln(x))' = 3x^2 \ln(x) + x^3 \ln(x)$

Solutions

1(c) $\int_1^3 3x^2 \ln(x) + x^3 \left(\frac{1}{x}\right)\ dx$

- **Goal 1:** find an antiderivative $F(x)$ of $3x^2 \ln(x) + x^3 \left(\frac{1}{x}\right)$.
  - **Conclusion:** $F(x) = x^3 \ln(x)$.
- **Goal 2:** find the value of the definite integral
  Using FTC v2,

$$\begin{aligned} \int_1^3 3x^2 \ln(x) + x^3 \left(\frac{1}{x}\right)\ dx &= x^3 \ln(x)\Big|_1^3 \\ &= 3^3 \ln(3) - 1^3 \ln(1) \\ &= 27 \ln(3). \end{aligned}$$

1(d) $\int_0^1 x^{12}e^{x^{13}}\ dx$

- **Goal 1:** find an antiderivative $F(x)$ of $x^{12}e^{x^{13}}$.
  - $x^{12}e^{x^{13}}$ is a product. A function can differentiate into a product if the original function – the one we're trying to find, the one that differentiates to what we have – is a product or a composition.
  - F is unlikely to itself be a product, since the product rule produces a sum of two terms, and here we only have a single term.
  - F is most likely a composition!
  - Looking more closely, there is indeed already a composition present: $e^{x^{13}}$.
  - The product that comes from the chain rule is $[f(u)]' = f'(u)u'$. The composition is $e^{x^{13}}$. The "inside" of that is $u = x^{13}$, in which case $u' = 13x^{12}$.
  - *Try:* $F(x) = e^{x^{13}}$.
  - *Check:* $F'(x) = e^{x^{13}} \cdot 13x^{12}$. Not quite what we started with – 13 times what we want.
  - *Try:* $F(x) = \frac{1}{13}e^{x^{13}}$.
  - *Check:* $F'(x) = \frac{1}{13}e^{x^{13}} \cdot 13x^{12} = x^{12}e^{x^{13}}$

1(d) $\int_0^1 x^{12} e^{x^{13}}\, dx$

- **Goal 1:** Find an antiderivative $F(x)$ of $x^{12} e^{x^{13}}$.

$$F(x) = \frac{1}{13} e^{x^{13}}.$$

- **Goal 2:** find the value of the definite integral
  Using FTC v2,

$$\int_0^1 x^{12} e^{x^{13}}\, dx = \frac{1}{13} e^{x^{13}} \Big|_0^1 = \frac{1}{13} e^{1^{13}} - \frac{1}{13} e^{0^{13}} = \frac{1}{13}(e-1).$$

1(e) $\int_0^3 4e^x x + 2e^x x^2 \, dx$

- **Goal 1:** find an antiderivative $F(x)$ of $4e^x x + 2e^x x^2$.
  - Can't antidifferentiate either product on its own.
  - The product rule produces a sum of products of the form $(uv)' = uv' + u'v$.
  - One of the factors in each product is e^x (which could be both u and u'). The remaining factors in the two products are $2x^2$ and $4x$. If we let $v = 2x^2$, then v' must be $4x$, and we have that

    $$f(x) = uv' + u'v,$$

    so an antiderivative should be

    $$F(x) = uv = 2x^2 e^x.$$

  - *Check:* $F'(x) = 2x^2 e^x + 4xe^x$, which is what we started with!
  - *Conclusion:* $F(x) = 2x^2 e^x$.
- **Goal 2:** find the value of the definite integral
  Using FTC v2,

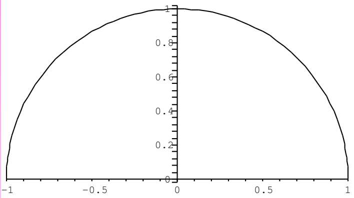
$$\int_0^3 4e^x x + 2e^x x^2 \, dx = 2x^2 e^x \Big|_0^3 = 2(3)^2 e^3 - 2(0)^2 e^0 = 2(9)e^3 - 0 = 18e^3.$$

1(f) $\int_{-1}^{1} \sqrt{1-x^2}\, dx$

- **Goal 1:** find an antiderivative $F(x)$ of $\sqrt{1-x^2}$.
  - Whatever F might be, it needs to differentiate into the above composition.
  - That suggests that my original function needs to also be a composition.
  - When I use the chain rule to differentiate a composition, the inside stays the same, and then I multiply by the derivative of the inside.
  - My *integrand* is not a product.
  - Uh oh! We're not going to be able to find an antiderivative anytime soon, if ever!
- **New Goal 1:** think about this differently!
  The FTC, v2, isn't going to help us this time. But because $\sqrt{1-x^2}$ is continuous on the interval $[-1, 1]$, we know from the FTC, v1, that this signed area exists – we can also just look at the graph to see that the signed area exists. So ... let's look at the graph and see if it's helpful!

1(f) $\int_{-1}^{1} \sqrt{1-x^2}\, dx$

- **New Goal 1:** Figure out the signed area by using a graph



By **definition** of the definite integral, the value of this definite integral is simply the signed area between the curve and the x-axis from -1 to 1.

Looking at this, we can see that this is just the area of a semi-circle of radius 1.
Therefore,

$$\int_{-1}^{1} \sqrt{1-x^2}\, dx = \text{half the area of a circle of radius } 1 = \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}.$$

2. Let $f(t) = 2t\cos(t^2)$ and $F(x) = \int_1^x f(t)\ dt$.

(a) Find the equation of the line tangent to $y = F(x)$ at $x = 3$.

Need: a point on the line, the slope of the line.

- **Slope of the tangent line at $x = 3$: $F'(3)$.**

  FTC, form 1 $\Rightarrow F'(x) = \frac{d}{dx}\left(\int_1^x 2t\cos(t^2)\ dt\right) = 2x\cos(x^2)$.

  Thus $F'(3) = 6\cos(9) \approx -5.47$.

2.(a) Let $f(t) = 2t\cos(t^2)$ and $F(x) = \int_1^x f(t)\,dt$. Find the equation of the line tangent to $y = F(x)$ at $x = 3$.

- **Slope of the tangent line:** $F'(3) = 6\cos(9) \approx -5.47$.
- **Point on the tangent line:** the point of tangency $(3, F(3))$.

$$F(3) = \int_1^3 f(t)\,dt = \int_1^3 2t\cos(t^2)\,dt.$$

Need an antiderivative of the product $2t\cos(t^2)$.

Probably came from a composition, so compare to $\left[f(u)\right]' = f'(u)u'$.
If $u = t^2$, $u' = 2t$, and $f'(u) = \cos(u)$ so $f(u) = \sin(u)$.

Try: F(x)=$\sin(t^2)$ *Check:* $\dfrac{d}{dt}(\sin(t^2)) = \cos(t^2) \cdot 2t$

Therefore the FTC, v2 tells me that

$$F(3) = \sin(t^2) \text{ from } 1to3 = \sin(9) - \sin(1) \approx -0.43.$$

2.(a) Let $f(t) = 2t\cos(t^2)$ and $F(x) = \int_1^x f(t)\,dt$. Find the equation of the line tangent to $y = F(x)$ at $x = 3$.

- **Slope of the tangent line:** $F'(3) = 6\cos(9) \approx -5.47$
- **Point on the tangent line:** $(3, -0.43)$
- **Equation of the tangent line:**

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - (\sin(9) - \sin(1)) &= 6\cos(9)(x - 3) \\ y + .43 &\approx -5.47(x - 3) \end{aligned}$$

2. Let $f(t) = 2t\cos(t^2)$ and $F(x) = \int_1^x f(t)\, dt$.

(b) Find a formula for $\frac{d}{dx}\left(F(x^3)\right)$.
Let $G(x) = F(x^3) = F(u)$, where $u(x) = x^3$.
Then the chain rule tells us that

$$G'(x) = F'(u)u'(x) = F'(u) \cdot 3x^2.$$

We know from the FTC, v1, that $F'(x) = f(x) = 2x\cos(x^2)$, so $F'(u) = 2u\cos(u^2) = 2x^3\cos(x^6)$.
Therefore

$$\frac{d}{dx}(F(x^3)) = [2x^3\cos(x^6)] \cdot (3x^2) = 6x^5\cos(x^6).$$

In this particular case, because we could actually antidifferentiate $2x\cos(x^2)$, we could have done this another way, but I wanted to demonstrate this way, as it works even when we can't antidifferentiate the integrand.