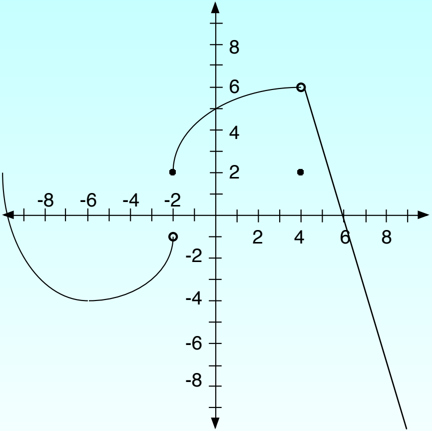
Solutions to In Class Work

Consider the function $f(x)$ defined by the graph below. Find the following:



1. $f(4) = y$-value of solid circle $= 2$:
2. $\lim_{x \to 4^+} f(x) = 6$, $\lim_{x \to 4^-} f(x) = 6$.

   Remember: for limits, we do not pay any attention to what happens **at** $x = 4$.
3. $\lim_{x \to 4} f(x) = 6$, since both the left- and right-limits were 6.

   Notice: $\lim_{x \to 4} f(x) \neq f(4)$.
4. Can see f is not continuous at $x = 4$.

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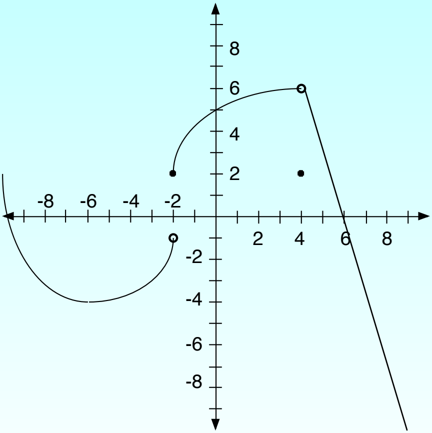

5. $f(-2) = y$-value of closed circle $=$ 2

6. $\lim_{x \to -2^+} f(x) = 2$,
$\lim_{x \to -2^-} f(x) = -1$

7. $\lim_{x \to -2} f(x)$ d.n.e. (does not exist), because the left- and right- sided limits differ.

   Notice: As with $x = 4$, $\lim_{x \to -2} f(x) \neq f(-2)$

8. Can see f isn't continuous at $x = -2$.

Solutions to In Class Work

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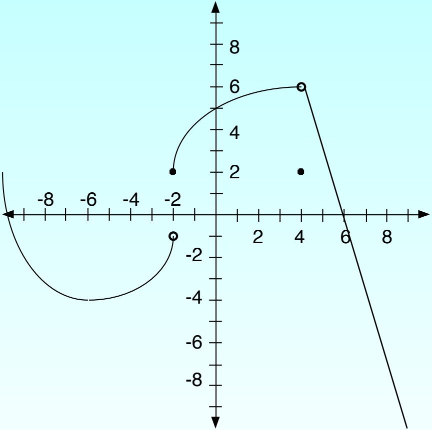
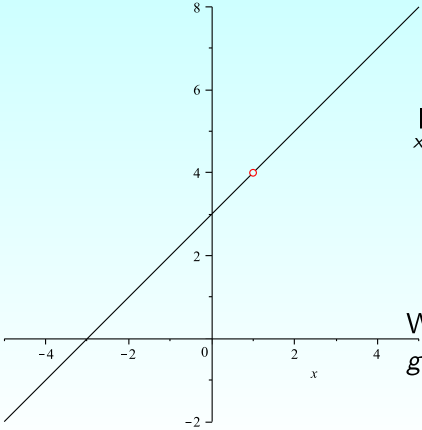


9. $f(-6) = -4$
10. $\lim_{x \to -6^+} f(x) = -4$,
    $\lim_{x \to -6^-} f(x) = -4$
11. $\lim_{x \to -6} f(x) = -4$
    Notice: $\lim_{x \to -6} f(x) = f(-6)$
12. Can see f *is* continuous at $x = 6$

Reading Question # 4 From Wednesday (Example 1.4.2)

Let $g(x) = \begin{cases} \dfrac{x^2+2x-3}{x-1} & \text{if } x \neq 1, \\ a & \text{if } x = 1 \end{cases}$ for some real number a.



$$\begin{aligned} \lim_{x\to 1} g(x) &= \lim_{x\to 1} \frac{x^2+2x-3}{x-1} \\ &= \lim_{x\to 1} \frac{(x+3)(x-1)}{x-1} \\ &= \lim_{x\to 1} (x+3) = 4 \end{aligned}$$

Will $g(x)$ be continuous if we define $g(1) = 7$?

Reading Question # 5 From Wednesday

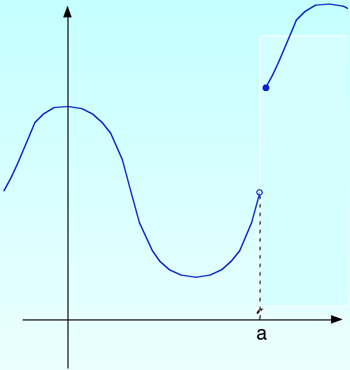


Figure 1.22b

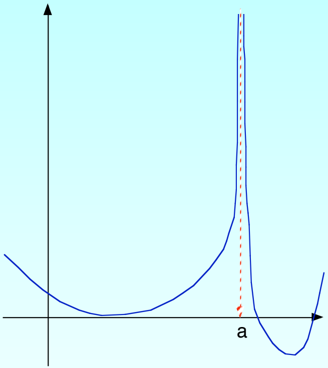


Figure 1.22d

Introductory Rules for Limits

Let a and c be any constants, and suppose that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist. Then:

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Introductory Rules for Limits

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- $\lim_{x\to a} p(x) = p(a)$,
- $\lim_{x\to a} \left(f(x)\right)^{1/n} = \left(\lim_{x\to a} f(x)\right)^{1/n}$, as long as the root makes sense.

Limits you will find helpful:

For any real number a, we have:

(i) $\lim_{x \to a} \sin(x) = \sin(a)$

(ii) $\lim_{x \to a} \cos(x) = \cos(a)$

(iii) $\lim_{x \to a} e^x = e^a$

(iv) $\lim_{x \to a} \ln(x) = \ln(a)$, for $a > 0$

(v) If $p(x)$ is a polynomial, then $\lim_{x \to a} f\left(p(x)\right) = \lim_{x \to p(a)} f(x)$.

(vi) $\lim_{x \to 0} \dfrac{\sin(x)}{x} = 0$

Recall:

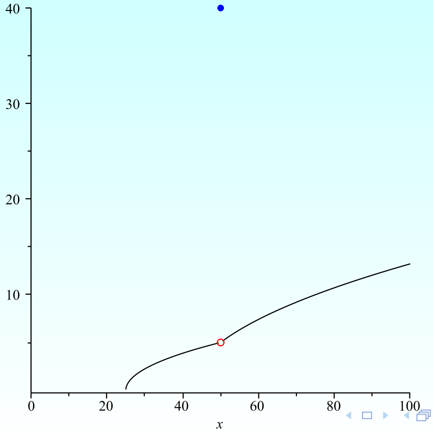
Definition: A function $f(x)$ is **continuous** at $x = a$ if

- $\lim_{x \to a} f(x)$ exists
- $f(a)$ is defined
- $\lim_{x \to a} f(x) = f(a)$

More concisely:
A function $f(x)$ is **continuous** at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

Graph of function for Example:

$$f(x) = \begin{cases} \sqrt{x-25} & \text{if } x < 50 \\ 40 & \text{if } x = 50 \\ \sqrt{3x-125} & \text{if } x > 50 \end{cases}$$



In Class Work

1. $\lim_{x \to 0} (x^2 - 3x + 1)$

2. $\lim_{x \to 2} \dfrac{x^2 - x - 2}{x^2 - 4}$

3. $\lim_{h \to 0} \dfrac{(2+h)^2 - 4}{h}$

4. $\lim_{x \to -1} f(x)$, where $f(x) = \begin{cases} 2x + 1 & \text{if } x < -1 \\ 3 & \text{if } -1 < x < 1 \\ 2x + 1 & \text{if } x > 1 \end{cases}$

5. If $\lim_{x \to a} f(x) = 2$,
$\lim_{x \to a} g(x) = -3$,
& $\lim_{x \to a} h(x) = 0$,
determine the limits:
   (a) $\lim_{x \to a} \left[2f(x) - 3g(x)\right]$
   (b) $\lim_{x \to a} \left[3f(x)g(x)\right]$
   (c) $\lim_{x \to a} \left\{\dfrac{f(x) + g(x)}{h(x)}\right\}$
   (d) $\lim_{x \to a} \left\{\dfrac{3f(x) + 2g(x)}{h(x)}\right\}$

Solutions:

1. $\lim_{x\to 0}(x^2 - 3x + 1)$
   Because $x^2 - 3x + 1$ is a polynomial, we know that

   $$\lim_{x\to a} p(x) = p(a) \textit{(Find limit of poly at a by plugging in a)}$$

   or in this case,

   $$\lim_{x\to 0}(x^2 - 3x + 1) = 0^2 - 3 \cdot 0 + 1 = 1.$$

Solutions:

2. $\lim_{x\to 2} \dfrac{x^2 - x - 2}{x^2 - 4}$

Know: $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{f(a)}{g(a)}$, if $\lim_{x\to a} g(x) \neq 0$.

But: $\lim_{x\to 2}(x^2 - 4)$ is 0, so can't use that rule

Factor!

$$\lim_{x\to 2} \frac{x^2 - x - 2}{x^2 - 4} = \lim_{x\to 2} \frac{(x-2)(x+1)}{(x-2)(x+2)}.$$

As x approaches 2, we're not ever letting x be 2, so $x - 2$ won't be 0, and so we can cancel the common factor.

$$\lim_{x\to 2} \frac{x^2 - x - 2}{x^2 - 4} = \lim_{x\to 2} \frac{(x-2)(x+1)}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x+1}{x+2} = \frac{\lim_{x\to 2}(x+1)}{\lim_{x\to 2}(x+2)} = \frac{3}{4}.$$

Solutions:

3. $\lim_{h\to 0} \dfrac{(2+h)^2-4}{h}$

Can't use that $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{f(a)}{g(a)}$, since the denominator approaches 0.

Expand numerator, see if we can come up with anything!

$$\lim_{h\to 0} \frac{(2+h)^2-4}{h} = \lim_{h\to 0} \frac{4+4h+h^2-4}{h} = \lim_{h\to 0} \frac{4h+h^2}{h}.$$

Again, because h is approaching but never reaching 0 (the place where the denominator is 0), we can cancel the common factor of h:

$$\lim_{h\to 0} \frac{(2+h)^2-4}{h} = \lim_{h\to 0}(4+h) = 4.$$

Solutions:

4. $\lim_{x \to -1} f(x)$, where $f(x) = \begin{cases} 2x+1 & \text{if } x < -1 \\ 3 & \text{if } -1 < x < 1 \\ 2x+1 & \text{if } x > 1 \end{cases}$

$\lim_{x \to -1^{-1}} f(x) = \lim_{x \to -1^-} 2x + 1 = 2(-1) + 1 = -1$, since $2x + 1$ is a polynomial.

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} 3 = 3$, because the limit of a constant is that constant.

Since the left- and right-sided limits don't agree, $\lim_{x \to -1} f(x)$ does not exist (even though $f(-1) = 3$).

Solutions:

5. If $\lim_{x\to a} f(x) = 2$, $\lim_{x\to a} g(x) = -3$, & $\lim_{x\to a} h(x) = 0$, determine the limits:

   (a) $\lim_{x\to a} \left[2f(x) - 3g(x)\right]$

$$\begin{aligned} \lim_{x\to a} \left[2f(x) - 3g(x)\right] &= 2\lim_{x\to a} f(x) - 3\lim_{x\to a} g(x) \\ &= 2(2) - 3(-3) = 4 + 9 = 13. \end{aligned}$$

   (b) $\lim_{x\to a} \left[3f(x)g(x)\right]$

$$\begin{aligned} \lim_{x\to a} \left[3f(x)g(x)\right] &= 3\lim_{x\to a} f(x) \lim_{x\to a} g(x) \\ &= 3(2)(-3) = -18. \end{aligned}$$

5. (continued) If $\lim_{x \to a} f(x) = 2$, $\lim_{x \to a} g(x) = -3$, & $\lim_{x \to a} h(x) = 0$, determine the limits:

(c) $\lim_{x \to a} \left\{ \dfrac{f(x) + g(x)}{h(x)} \right\}$

- Numerator: $\lim_{x \to a} \left(f(x) + g(x) \right) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = 2 - 3 = -1$
- Denominator: $\lim_{x \to a} h(x) = 0$

What happens when the numerator is approaching -1 and the denominator approaches 0?

To get a feel for this, consider what happens when the numerator **is fixed** at -1 while the denominator approaches 0 from the positive side:

$$\frac{-1}{2} = -0.5 \qquad \frac{-1}{1} = -1 \qquad \frac{-1}{0.1} = -10$$

$$\frac{-1}{0.01} = -100 \qquad \frac{-1}{0.001} = -1000 \qquad \frac{-1}{0.0001} = -10000$$

In general, if the numerator approaches a finite non-zero number while the denominator approaches 0, the limit does not exist.

5. (continued) If $\lim_{x \to a} f(x) = 2$, $\lim_{x \to a} g(x) = -3$, & $\lim_{x \to a} h(x) = 0$, determine the limits:

(d) $\lim_{x \to a} \left\{ \dfrac{3f(x) + 2g(x)}{h(x)} \right\}$

- Numerator:
  $\lim_{x \to a} \left(3f(x) + 2g(x)\right) = 3 \lim_{x \to a} f(x) + 2 \lim_{x \to a} g(x) = 3(2) + (2)(-3) = 0$
- Denominator $\lim_{x \to a} h(x) = 0$

In this case, we don't know what happens – this limit is in **indeterminate form**, and evaluating it (at this point) is not possible.