In Class Work

1. Use the limit definition to find the slope of the line tangent to $f(x) = x^2 - 3x$ at the following points.

   If you happen to have learned a short cut for this process in a past class, do NOT use it here. The point of this work is to practice with and understand the limit definition of the slope of the secant line.

   (a) $a = 0$

   (b) $a = -2$

   (c) An unspecified point $a = x$

2. Find the equation of the line tangent to $f(x) = x^2 - 3x$ at $a = -2$.

Solutions

1. Use the limit definition to find the slope of the line tangent to $f(x) = x^2 - 3x$ at:

   (a) $a = 0$

$$\begin{aligned} m_{tan} &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{\left[(h)^2 - 3(h)\right] - \left[0^2 - 3(0)\right]}{h} \\ &= \lim_{h\to 0} \frac{h^2 - 3h}{h} \\ &= \lim_{h\to 0} h - 3 \\ &= -3 \end{aligned}$$

Solutions

1. Use the limit definition to find the slope of the line tangent to $f(x) = x^2 - 3x$ at the following points.

   (b) $a = -2$

$$
\begin{aligned}
m_{tan} &= \lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h} \\
&= \lim_{h\to 0} \frac{\left[(-2+h)^2 - 3(-2+h)\right] - \left[2^2 - 3(-2)\right]}{h} \\
&= \lim_{h\to 0} \frac{\left(4 - 4h + h^2 + 6 - 3h\right) - \left(4+6\right)}{h} \\
&= \lim_{h\to 0} \frac{-4h + h^2 - 3h}{h} = \lim_{h\to 0} -4 + h - 3 \\
&= -7
\end{aligned}
$$

Solutions

1. Use the limit definition to find the slope of the line tangent to $f(x) = x^2 - 3x$ at:

   (c) An unspecified point $a = x$

$$\begin{aligned}
m_{tan} &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\left[(x+h)^2 - 3(x+h)\right] - \left[x^2 - 3(x)\right]}{h} \\
&= \lim_{h\to 0} \frac{\left(x^2 + 2xh + h^2 - 3x - 3h\right) - \left(x^2 - 3x\right)}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2 - 3h}{h} \\
&= \lim_{h\to 0} 2x + h - 3 \\
&= 2x - 3
\end{aligned}$$

Solutions

2. Find the equation of the line tangent to $f(x) = x^2 - 3x$ at $a = -2$.

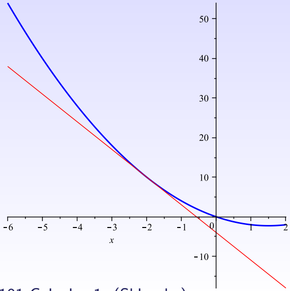
   Need: point and a slope

   Slope: $m_{tan}(-2) = -7$

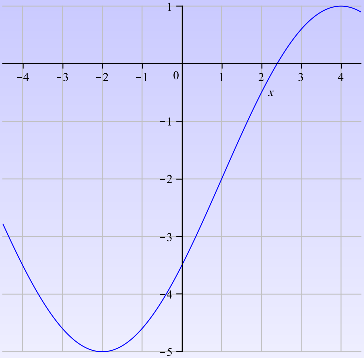
   Point: Use point of tangency, $\big(-2, f(-2)\big) = (-2, 4+6) = (-2, 10)$

   Thus the tangent line is given by

   $$y - 10 = -7(x+2) \Longrightarrow y = -7x - 4.$$



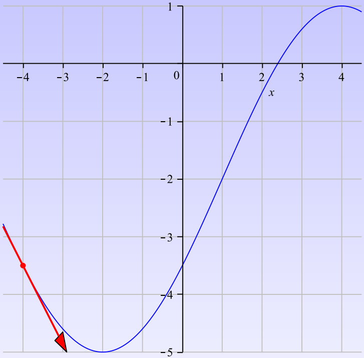
In Class Work



Consider the function $f(x)$ shown in the graph.

1. At $x = -4$:
   (a) is f increasing or decreasing?
   (b) is the inst. rate of change > 0 ?
   (c) is $m_{tan} > 0$?
2. For what x is the inst. rate of change of f zero?
3. At $x = 3$:
   (a) is f increasing faster and faster, or slower and slower?
   (b) is the rate of change increasing or decreasing?
   (c) is m_{tan} increasing or decreasing?
4. For what x does the slope of f reach a maximum?
5. Roughly sketch the graph of m_{tan}

Solutions:



Consider the function $f(x)$ shown in th graph.

1. At $x = -4$:
   (a) is f increasing or decreasing?

   f is decreasing at $x = -4$

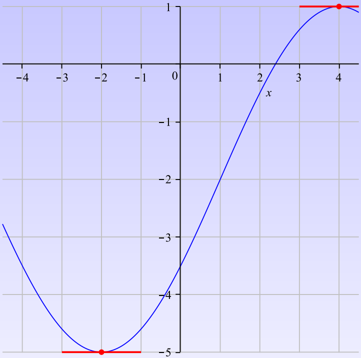
   (b) is the rate of change > 0 ?

   No: because f is decreasing at $x = -4$, the rate of change is negative.

   (c) is $m_{tan} > 0$?

   No: because the function is decreasing at $x = -4$, it is sloping downward.

Solutions



Consider the function $f(x)$ shown in the graph.

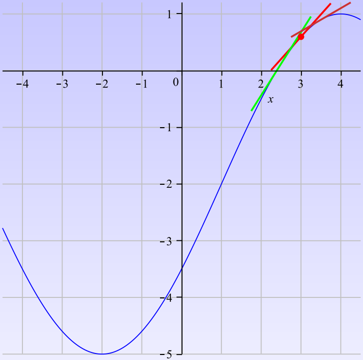
2. For what x is the rate of change of f zero?

   The points where the rate of change is 0 are the points where f isn't changing – neither increasing nor decreasing.

   That is, they are the points where the slope of the tangent line is 0: the points where the graph is flat.

   These points occur at $x = -2$ and $x = 4$.

Solutions



Consider the function $f(x)$ shown in the graph.

3. At $x = 3$:
   (a) is f increasing faster and faster, or slower and slower?

   At $x = 3$, f is increasing, but it's slowing down.

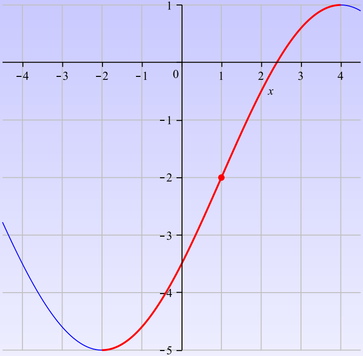
   (b) is the rate of change increasing or decreasing?

   At $x = 3$, the rate of change is decreasing.

   (c) is m_{tan} increasing or decreasing?

   At $x = 3$, m_{tan} is decreasing.

Solutions



Consider the function $f(x)$ shown in the graph.

4. For what x does the slope of f reach a maximum?

   At $x = -2$, $m_{tan} = 0$.

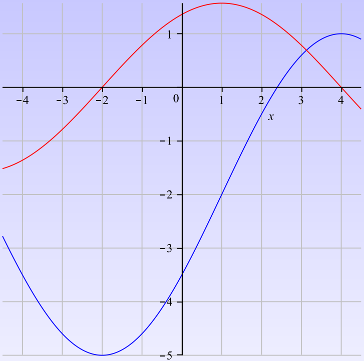
   As we move to the right, the slope starts out small but increases.

   That continues until we reach about $x = 1$ (the **inflection point**).

   After $x = 1$, the the slope decreases until it reaches 0 again at $x = 4$.

   Thus the slope reaches a maximum at $x = 1$, roughly.

Solutions



Consider the function $f(x)$ shown in the graph.

5. Roughly sketch the graph of m_{tan}

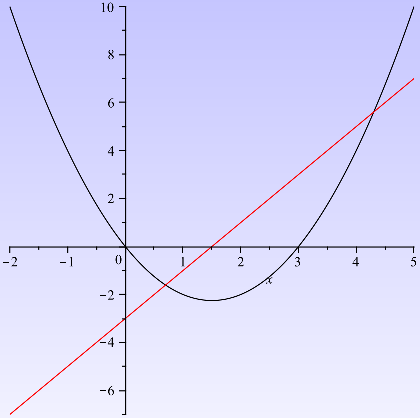
   $m_{tan} > 0$ on $(-2, 4)$, $m_{tan} < 0$ on $[-4.5, -2)$ and $(4, 4.5]$

   m_{tan} is increasing from 0 to its maximum on $[-2, 1)$ and then decreasing to 0 on $(1, 4]$.

   From $x = -4$ to $x = -2$, m_{tan} is going from being very negative to being closer and closer to 0, so it's increasing.

   From $x = 4$ to $x = 4.5$, m_{tan} is going from being 0 to being more and more negative, so it's decreasing.

Graphs of $f(x) = x^2 - 3x$ and the function that gives the slope of f at any point x, which we found to be $2x - 3$.



Notice that $f(x)$ is decreasing up until about $x = 1.5$ and that the graph that shows its slope is negative over the same interval. At $x = 1.5$, $f(x)$ is flat – that is, it isn't changing, or its tangent line is horizontal – and the graph of its slope is 0 there. And finally, to the right of $x = 1.5$, the graph of $f(x)$ is increasing, and the graph of its slope is positive.

In Class Work

1. Use the limit definition to find the slope of the line tangent to $f(x)$ at an unspecified point x:
   (a) $f(x) = 2x$
   (b) $f(x) = x^2$
   (c) $f(x) = 5$

Solutions

1. Use the limit definition to find the slope of the line tangent to $f(x)$ at an unspecified point x:

   (a) $f(x) = 2x$

$$\begin{aligned} m_{tan} &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{2(x+h)-2x}{h} \\ &= \lim_{h\to 0} \frac{2h}{h} = \lim_{h\to 0} 2 \\ &= 2 \end{aligned}$$

Thus the slope of the tangent line is always 2, no matter what x we're looking at.

In other words, the graph of $2x$ has a constant slope of 2.

Question: What does having a constant slope tell us about the graph of $f(x)$?

Solutions

1. Use the limit definition to find the slope of the line tangent to $f(x)$ at an unspecified point x:

   (b) $f(x) = x^2$

$$\begin{aligned} m_{tan} &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h} \\ &= \lim_{h\to 0} (2x+h) \\ &= 2x \end{aligned}$$

So now we know that the graph of x^2 has a slope of 4 at $x = 2$, a slope of 2 at $x = 1$, a slope of -2 at $x = -1$, etc.

Solutions

1. Use the limit definition to find the slope of the line tangent to $f(x)$ at an unspecified point x:

    (c) $f(x) = 5$

$$\begin{aligned} m_{tan} &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(5)-(5)}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0 \end{aligned}$$