Recall:

Let $f(x)$ be a function.

The **derivative** of f at x is the slope of the line tangent to f at any point x … which is the same as the instantaneous rate of change of f at any point x.

We represent the derivative of f at x by $f'(x)$

Definition: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

The process of computing the derivative function is called **taking the derivative**, or *differentiating*.

Summary: Derivatives Found Using Limit Definition

$$f(x) = 5 \implies f'(x) = 0 \text{ (Fri 9/23)}$$
$$f(x) = 2x \implies f'(x) = 2 \text{ (Fri 9/23)}$$
$$f(x) = x^2 \implies f'(x) = 2x^1 \text{ (Fri 9/23)}$$
$$f(x) = x^{1/2} \implies f'(x) = \frac{1}{2}x^{-1/2} \text{ (Reading for Fri)}$$

$$f(x) = x^2 - 3x \implies f'(x) = 2x - 3 \text{ (Th 9/22)}$$
$$f(x) = 2x^2 - 3x \implies f'(x) = 4x - 3 \text{ (RQ for Fri 9/23)}$$
$$f(x) = 3x^3 + 2x - 1 \implies f'(x = 9x^2 + 2 \text{ (Reading for F)}$$

Alternative Notation for the Derivative:

If I want you to take the derivative of $x^5 - 7x^4 + 15x^3 - 12x^2 + 42x - 16$, I could write:

Find $\left(x^5 - 7x^4 + 15x^3 - 12x^2 + 42x - 16\right)'$.

But the actual mathematical instruction comes at the very end that way.

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If I want you to take the derivative of $x^5 - 7x^4 + 15x^3 - 12x^2 + 42x - 16$, I could write:

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But the actual mathematical instruction comes at the very end that way.

Instead, use Leibniz notation:
Find $\dfrac{d}{dx}\left(x^5 - 7x^4 + 15x^3 - 12x^2 + 42x - 16\right)$

Solutions to Friday's In Class Work

2. Use the limit definition of the derivative to find the derivative of the following:
   (d) $f(x) = x^3$
   Time-saver: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h\to 0} \frac{\left(x^3 + 3x^2h + 3xh^2 + h^3\right) - x^3}{h} \\ &= \lim_{h\to 0} \left(3x^2 + 3xh + h^2\right) \\ &= 3x^2 \end{aligned}$$

Solutions to Friday's In Class Work

2. (e) $f(x) = x^{-1} = \dfrac{1}{x}$

   Remember common denominators

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{x+h} - \frac{1}{x}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\frac{x-(x+h)}{x(x+h)} \\
&= \lim_{h\to 0} \frac{-h}{xh(x+h)} \\
&= \lim_{h\to 0} \frac{-1}{x(x+h)} \\
&= -\frac{1}{x^2} \\
&= -x^{-2}
\end{aligned}
$$

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$$f(x) = 2x \implies f'(x) = 2 \text{ (Fri 9/23)}$$

$$f(x) = x^2 \implies f'(x) = 2x^1 \text{ (Fri 9/23)}$$

$$f(x) = x^3 \implies f'(x) = 3x^2 \text{ (Fri 9/23)}$$

$$f(x) = x^{-1} \implies f'(x) = -x^{-2} \text{ (Fri 9/23)}$$

$$f(x) = x^{1/2} \implies f'(x) = \frac{1}{2}x^{-1/2} \text{ (Reading for Fri)}$$

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Recall: Derivatives Found Using Limit Definition

$$
\begin{aligned}
f(x) = 5x^0 &\implies f'(x) = 0 \text{ (Fri 9/24)}\\
f(x) = 2x^1 &\implies f'(x) = 2(1) \text{ (Fri 9/24)}\\
f(x) = x^2 &\implies f'(x) = 2x^1 \text{ (Fri 9/24)}\\
f(x) = x^3 &\implies f'(x) = 3x^2 \text{ (Fri 9/24)}\\
f(x) = x^{-1} &\implies f'(x) = -x^{-2} \text{ (Fri 9/24)}\\
f(x) = x^{1/2} &\implies f'(x) = \frac{1}{2}x^{-1/2} \text{ (Reading for Fri)}\\
f(x) = x^2 - 3x^1 &\implies f'(x) = 2x - 3(1) \text{ (Th 9/23)}\\
f(x) = 2x^2 - 3x^1 &\implies f'(x) = 2(2)x^1 - 3(1) \text{ (RQ for Fri 9/24)}\\
f(x) = 3x^3 + 2x^1 - 1(x^0) &\implies f'(x) = 3(3)x^2 + 2(1) + 0 \text{ (Reading for F)}
\end{aligned}
$$

Recall: Derivatives Found Using Limit Definition

$$
\begin{aligned}
f(x) = 5x^0 &\implies f'(x) = 0 \text{ (Fri 9/24)} \\
f(x) = 2x^1 &\implies f'(x) = 2(1) \text{ (Fri 9/24)} \\
f(x) = x^2 &\implies f'(x) = 2x^1 \text{ (Fri 9/24)} \\
f(x) = x^3 &\implies f'(x) = 3x^2 \text{ (Fri 9/24)} \\
f(x) = x^{-1} &\implies f'(x) = -x^{-2} \text{ (Fri 9/24)} \\
f(x) = x^{1/2} &\implies f'(x) = \frac{1}{2}x^{-1/2} \text{ (Reading for Fri)} \\
f(x) = x^2 - 3x^1 &\implies f'(x) = 2x - 3(1) \text{ (Th 9/23)} \\
f(x) = 2x^2 - 3x^1 &\implies f'(x) = 2(2)x^1 - 3(1) \text{ (RQ for Fri 9/24)} \\
f(x) = 3x^3 + 2x^1 - 1(x^0) &\implies f'(x) = 3(3)x^2 + 2(1) + 0 \text{ (Reading for F)}
\end{aligned}
$$

Conjectures: Looks as if $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$,

$\frac{d}{dx}\left(f(x) \pm g(x)\right) = f'(x) \pm g'(x)$, and $\frac{d}{dx}\left(cf(x)\right) = cf'(x)$

Are the Conjectures True?

1. $\frac{d}{dx}\Big(f(x) \pm g(x)\Big) = f'(x) \pm g'(x)$ - follows directly from limit rules:

$$\begin{aligned}\frac{d}{dx}\Big(f(x) \pm g(x)\Big) &= \lim_{h\to 0} \frac{\big(f(x+h) \pm g(x+h)\big) - \big(f(x) \pm g(x)\big)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \pm \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= f'(x) \pm g'(x)\end{aligned}$$

2. $\frac{d}{dx}\Big(cf(x)\Big) = cf'(x)$ - also follows directly from limit rules

$$\begin{aligned}\frac{d}{dx}\Big(cf(x)\Big) &= \lim_{h\to 0} \frac{cf(x+h)-cf(x)}{h} \\ &= \lim_{h\to 0} \frac{c\big(f(x+h)-f(x)\big)}{h} \\ &= c\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = cf'(x)\end{aligned}$$

3. How about $\dfrac{d}{dx}\left(x^n\right) = nx^{n-1}$?

Remember: Pascal's Triangle

$(x+h)^0$
$(x+h)^1$
$(x+h)^2$
$(x+h)^3$
$(x+h)^4$
$(x+h)^5$

Remember: Pascal's Triangle

$(x+h)^0$	1
$(x+h)^1$	$x+h$
$(x+h)^2$	$x^2+2xh+h^2$
$(x+h)^3$	$x^3+3x^2h+3xh^2+h^3$
$(x+h)^4$	$x^4+4x^3h+6x^2h^2+4xh^3+h^4$
$(x+h)^5$	

Remember: Pascal's Triangle

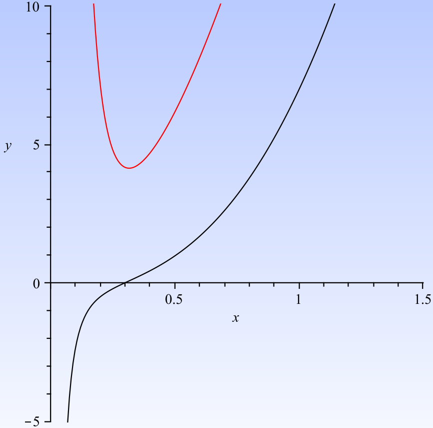
$(x+h)^0$	1	1
$(x+h)^1$	$x+h$	1 1
$(x+h)^2$	$x^2+2xh+h^2$	1 2 1
$(x+h)^3$	$x^3+3x^2h+3xh^2+h^3$	1 3 3 1
$(x+h)^4$	$x^4+4x^3h+6x^2h^2+4xh^3+h^4$	1 4 6 4 1
$(x+h)^5$		1 5 10 10 5 1

Properties of Derivatives We Know So Far:

- $\frac{d}{dx}\left(c\right) = 0$ -horizontal lines have slope 0 everywhere
- $\frac{d}{dx}\left(cf(x)\right) = cf'(x)$. $cf(x)$ grows c times faster than $f(x)$ does, so slope should be c times as big.
- $\frac{d}{dx}\left(f(x) \pm g(x)\right) = f'(x) \pm g'(x)$ - follows from definition of derivative, and fact that limit of sum=sum of limits
- $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$ - the Power Rule

Verifying a derivative

Graphs of $f(x) = 7x^e - \dfrac{1}{42x^2}$ and what we hope is in fact $f'(x)$:
$7ex^{e-1} + \frac{1}{21x^3}$



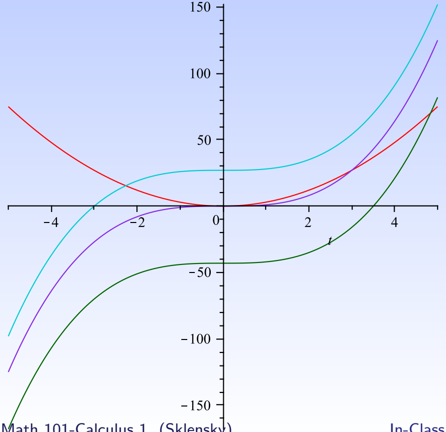
- $f(x)$ never flat
  $\implies$ Slope $f'(x)$ never 0
- $f(x)$ always $\uparrow$
  $\implies$ Slope $f'(x) > 0$
- $f(x)$ steep near $x = 0$
  $\implies$ Slope $f'(x)$ large near 0
- $f(x)$ becomes less & less steep until $x =\sim 0.3$
  $\implies$ Slope $f'(x) \downarrow$ to minimum at $x =\sim 0.3$.
- $f(x)$ then becomes steeper and steeper
  $\implies$ Slope $f'(x) \uparrow$ again.

Verifying three antiderivatives

Original function: $v(t) = p'(t) = 3t^2$

Possible antiderivatives:

- $p(t) = t^3 + 27$
- $p(t) = t^3$
- $p(t) = t^3 - 43$



- $p'(t) = 0$ at $t = 0 \Longrightarrow$ $p(t)$ is flat at $t = 0$
- $p'(t) > 0$ for all $t \neq 0 \Longrightarrow$ $p(t) \uparrow$ for all $t \neq 0$
- $p'(t)$ first $\downarrow$ then $\uparrow \Longrightarrow$ $p(t)$ is getting flatter than steeper (since $p'(t) > 0$)

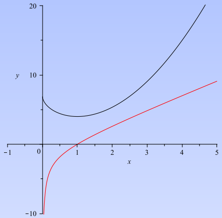
In Class Work

1. Find the derivatives of the following functions. If you have a graphing calculator, verify your answers by graphing f and f' on the same set of axes.
   (a) $f(x) = x^2 - 4x^{1/2} + 7$
   (b) $f(x) = x^3 - \frac{5}{x^2} + 2$
   (c) $f(x) = 2x^\pi + x^{-42} - 17x$
   (d) $f(x) = \frac{7}{x} - x + 4$
2. Find an *anti*derivative for each function in 1.

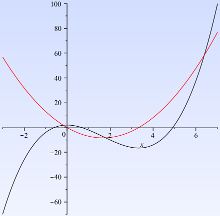
Solutions

1. Find the derivatives of the following functions.

(a) $f(x) = x^2 - 4x^{1/2} + 7$
$\Longrightarrow f'(x) = 2x - 2x^{-1/2}$



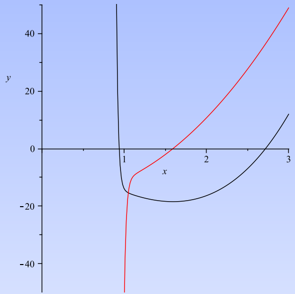
(b) $f(x) = x^3 - \dfrac{5}{x^2} + 2$
$\Longrightarrow f'(x) = 3x^2 + 10x^{-3}$



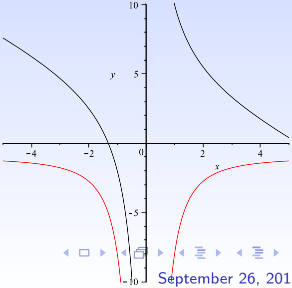
Solutions

1. Find the derivatives of the following functions.

(c) $f(x) = 2x^{\pi} + x^{-42} - 17x \Longrightarrow$
$f'(x) = 2\pi x^{\pi-1} - 42x^{-43} - 17$



$$f(x) = \frac{7}{x} - x + 4$$
$\Longrightarrow f'(x) = -7x^{-2} - 1$



Solutions

2. Find an *anti*derivative for each function in 1.

   (a) $f(x) = x^2 - 4x^{1/2} + 7 \Longrightarrow F(x) = x^3/3 - 4(\frac{2}{3})x^{3/2} + 7x$

   (b) $f(x) = x^3 - \dfrac{5}{x^2} + 2 \Longrightarrow F(x) = x^4/4 + 5/x + 2x$

   (c) $f(x) = 2x^{\pi} + x^{-42} - 17x \Longrightarrow$
   $F(x) = 2x^{\pi+1}/(\pi + 1) - x^{-41}/41 - 17x^2/2$

   (d) $f(x) = \dfrac{7}{x} - x + 4 \Longrightarrow F(x) =$ uh oh!

   When we try to follow our usual rule, it doesn't work!

   Does that mean that there *is* no function that antidifferentiates to $1/x$?

   Not necessarily – we've only looked at a very small collection of functions so far! But it does mean that no *power* function differentiates to be $1/x$!