Jill Grissom
4142 Sage Brush Street
Las Vegas, NV 89120
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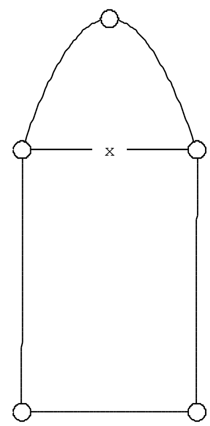
Math 101 Students
Wheaton College
Norton, MA 02766

Dear Calculus Students:

I've got the chance of a lifetime, but I need some help. You see, I've always wanted to be a Crime Scene Investigator like my uncle was before he retired. He got me an internship working with his former colleagues while I finish high school. A possible small-scale gambling fraud case came in, and I talked the bosses into giving me a couple of weeks to take a crack at solving it before they take over (if necessary). If I can get this solved, it'll surely help me get a real job investigating crime scenes when I finish college. The problem is, I'm stuck. Of course, if you do help me out, I'll give you full credit – I figure, even showing that I know when to ask for help and where to go to get it will look good.

The owner of a local pool hall (I'll call him "Moe") has been accused of running a betting scam in which the "house" (that is, "Moe") can never lose. My job is to determine whether this accusation is true. The hook to get people into the place is that there are many unusually-shaped pool tables – circles, ellipses, trapezoids, etc. In the center of the pool hall are two seemingly identical tables. Three sides of each of these two tables are the usual rectangular shape, but the fourth side has been replaced by a curving side bulging out. Where the curved part meets the two parallel sides is not especially smooth – it kind of makes a corner, or it would if there weren't pockets there. There are only five pockets in each table: the two I just mentioned where the curved side meets the two parallel sides, as well as two where the third straight side meets the two parallel sides, and one at the center of the curved side. Here's a rough sketch:



"Moe" stands at one of these tables in the middle and has a running challenge with his customers. He and the challenger take turns placing a cue ball where I've marked an x on the above sketch. The object is to get the ball to bounce right back over the spot marked x, without hitting the cushion along the edges more than once, and without the ball going into any of the pockets. (Of course, the most obvious shot- straight at the top of the curve- is out, as the ball would just fall into the pocket located there.) If successful, the other person pays up. If not successful, no money changes hands at all.

This being a betting town, and the challenge seeming so simple, people jump at the chance to challenge "Moe". The police have been checking "Moe" out for some time for other reasons, and noticed that the challengers *never* make the shot, while "Moe" makes the shot about half the time.

"Moe" 's getting rich fast – which is fine, as long as there's a legitimate chance the challengers can occasionally make the shot.

We've had people inspect the two tables, and we've discovered that as suspected they are *not* identical. There's a small tag on each – one says "parabolic model L" and the other says says "parabolic model G". Furthermore, while the rectangular portions of both are the same size – the playing surfaces of both rectangular portions are $4'6.0$" wide and $9.0'$ long (not including the cushions)– the curved portions differ slightly. On "Moe" 's table, the playing surface of the curved portion extends to a maximum distance of $1'10.0$" away from the spot marked x, at the center of the curved cushion. On the challenger's table, the curved portion extends to a maximum distance of only $1'6.0$" away from the spot marked x. The pockets on both tables are small – they're all 1.5" wide.

I've managed to sneak into "Moe" 's myself (despite being under 21). I didn't have enough money to risk challenging "Moe", but I did watch some of the challenges, and tried playing at some of the other oddly-shaped tables to see what I could observe about the ways the balls bounced. One thing I learned that might somehow be useful is that when I put my ball in the center of one of the *circular* tables, it didn't matter *where* I aimed the ball, it *always* bounced back over the center (except when I aimed at a pocket, of course). So there's obviously something fundamentally different between a circle and the curvy part of the parabolic tables.

Could you please prove to me, mathematically, whether or not this game is rigged? Is the 4" difference in the curved side enough to make it possible for "Moe" to make his shots while making it impossible for a standard pool player, not putting any fancy spin on the ball, to get his cue ball to bounce back over the x? Or is the game perfectly legitimate, and "Moe" 's just really good?

Also, these oddly shaped tables are proving so popular, I believe it's just a matter of time before this scam (if it is a scam) pops up again. It would be nice if we had a quick way to check in the future whether a two-parabolic-table set up like the one I just described is legitimate (with the challenger at least having *some* chance of winning), even if the tables have totally different dimensions from the two that Moe has. Could you please find out some quick way to measure a table and determine whether this challenge of shooting a cue ball from the spot marked x off the curved cushion and having it bounce back over its starting spot is possible? My ideal would be some sort of a test that anybody could do, without having to spend much time at all with a calculator. Maybe some ratios between a few key dimensions?

My supervisors have given me until **October 17th** to figure this out. Please send me a response (in the form of a letter to me) that can be presented in court as evidence by then.

Yours sincerely,
Jill Grissom

To: my Calc 1 students

A few thoughts that occurred to me as I quickly scanned through this letter that Jill sent to you, that may or may not prove useful:

- If you are not familiar with pool tables, you may want to look them up on the web before proceeding any further.
- Both pool tables are called *parabolic* models, so perhaps parabolas will play a role in your work. If they do, you may find it useful to remember that all upward- or downward-opening parabolas based at the origin have basically the same form: $y = cx^2$; in general, if the vertex of the parabola is at the point (h, k), then the form is $y - k = c(x - h)^2$. If you know the coordinates of any point on the parabola besides the vertex, you can always just plug that point in to find c.
- In a perfect world, if something hits a surface straight on (that is, if its path is perpendicular to the surface), it will bounce right back along the same path, but if it hits at any other angle, it won't – it will bounce off along a path that's at right angles to the one it came in on.
- On the circular pool table that Jill briefly refers to, if the ball is starting at the center, its path (before hitting the edge) is a radius of the circle. Perhaps I just have tangent lines on the brain because of the material we're covering right now, but I thought you might find it useful to know that any radius of any circle is perpendicular to the tangent line where the radius hits.
- Another random thought – you probably remember that two lines are perpendicular if their slopes are negative reciprocals of each other (that is, if one is m and the other is $-\frac{1}{m}$).
- Please take note that she is asking you to for a convincing mathematical argument as to whether the challenge is rigged. That means she's looking for calculations.
- Also please note that she asked you *two* questions: (1) is the game at Moe's pool hall rigged? and (2) can you come up for a quick test involving any two parabolic tables, no matter what their dimensions may be, to determine whether the challenge is possible or impossible?