Find the following integrals, and *check your answers!!*

1. $\displaystyle\int \frac{1}{\sqrt{1-x}}\,dx \qquad (u = 1-x)$

   Let $u = 1-x$. Then $\dfrac{du}{dx} = -1$, so $du = -1\ dx$, or $dx = -1\ du$. I of course chose u so you could substitute it directly into the integral – the question is, does du fit (whether directly or with a bit of manipulation) into the integral as well?

   Well, du doesn't equal dx exactly (if it did, this choice of u wouldn't simplify the integral at all!), but it's close.

   In the integral, we can replace $1-x$ with u and dx with $-1\ du$. When we do, we get the somewhat simpler integral

   $$\int \frac{1}{\sqrt{1-x}}\,dx = \int \frac{1}{\sqrt{u}} \cdot (-1)\ du.$$

   Notice that there are *no* terms involving x left – that's one key to a successful substitution.

   The other key is that the new integral must be simpler than the original, and I have to be able to antidifferentiate what's left!

   $$\begin{aligned} \int \frac{1}{\sqrt{u}} \cdot (-1)\ du &= -\int u^{-1/2}\ du \\ &= 2u^{1/2} + C \\ &= 2\sqrt{1-x} + C \end{aligned}$$

2. $\displaystyle\int x \sin(\pi x^2)\ dx \qquad (u = \pi x^2)$

   Let $u = \pi x^2$. Then $\dfrac{du}{dx} = 2\pi x$, so $du = 2\pi x\ dx$. I of course chose u so you could substitute it directly into the integral – the question is, does du fit (whether directly or with a bit of manipulation) into the integral as well?

   Looking at the original integral, I could rewrite it as

   $$\int \sin(\pi x^2) \cdot x\ dx.$$

With it written this way, I can see that this choice of u absorbs the πx^2, but du is more than I have left – the integral has an extra $x\ dx$, while du is $2\pi x\ dx$. These are essentially the same, as far as the x terms go, but they differ by a constant multiple:

$$\frac{1}{2\pi}du = x\ dx, \text{ which is what is left in the integral.}$$

Now we're ready to replace terms in the integral that involve x with equivalent terms that involve u: I'll replace πx^2 with u and $x\ dx$ with $\frac{1}{2\pi}\ du$.

$$\int x \sin(\pi x^2)\ dx = \int \sin(u) \cdot \frac{1}{2\pi}\ du.$$

Was this substitution successful? Remember, I need to have gotten rid of all of the x's – which I did. The other thing I need is of course to be able to antidifferentiate the new simpler integral. Can I? Let's try!

$$\begin{aligned}
\int x \sin(\pi x^2)\ dx &= \int \sin(u) \cdot \frac{1}{2\pi}\ du \\
&= \frac{1}{2\pi} \int \sin(u)\ du \\
&= \frac{1}{2\pi}(-\cos(u)) + C \\
&= -\frac{1}{2\pi}\cos(\pi x^2) + C
\end{aligned}$$

3. $\displaystyle\int_1^3 \frac{x}{1+x^2}\ dx \qquad (u = 1 + x^2)$

Let $u = 1 + x^2$. Then $\frac{du}{dx} = 2x$, so $du = 2x\ dx$. Again, we chose u so you could substitute it directly into the integral – but does du fit (whether directly or with a bit of manipulation) into the integral as well?

Looking at the original integral, I could rewrite it as

$$\int_1^3 \frac{x}{1+x^2}\ dx = \int_1^3 \frac{1}{1+x^2} \cdot x\ dx.$$

Once again, I can see that u absorbs the term $1+x^2$ in the denominator, but again, du is more than what's left over in the integral – left in the integral still is $x\ dx$, while we have that $du = 2x\ dx$. While they differ, though, they only differ by a constant multiple, which is easily dealt with:

$$\frac{1}{2}\ du = x\ dx, \text{ which is what I have leftover in my integral.}$$

Now we're ready to replace the terms involving x and dx in our original integral with equivalent terms involving u and du:

$$\int_1^3 \frac{x}{1+x^2}\ dx = \int_{x=1}^{x=3} \frac{1}{u} \cdot \frac{1}{2}\ du.$$

Was this a successful substitution? Well, we certainly got rid of all the x's. To see whether it made the integration possible, we dive right in:

$$\begin{aligned}\int_{x=1}^{x=3} \frac{1}{u} \cdot \frac{1}{2}\ du &= \frac{1}{2}\int_{x=1}^{x=3} \frac{1}{u}\ du \\ &= \frac{1}{2}(\ln(u)) \text{ from } x=1 \text{ to } x=3 \\ &= \frac{1}{2}\ln(1+x^2) \text{ from } x=1 \text{ to } x=3 \\ &= \frac{1}{2}[(\ln(1+3^2)) - \ln(1+1^2)] \\ &= \frac{1}{2}(\ln(10) - \ln(2))\end{aligned}$$

4. $\displaystyle\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\ dx \qquad (u = \sqrt{x+1})$

Let $u = \sqrt{x+1}$. Then $\dfrac{du}{dx} = \dfrac{1}{2}(x+1)^{-1/2} \cdot 1 = \dfrac{1}{2} \cdot \dfrac{1}{\sqrt{x+1}}$. Therefore $du = \dfrac{1}{2}\dfrac{1}{\sqrt{x+1}}\ dx$, so $2\ du = \dfrac{1}{\sqrt{x+1}}\ dx$.

Will this be a useful choice for u? Looking again at the original integral, I see I can rewrite it as

$$\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\ dx = \int e^{\sqrt{x+1}} \cdot \frac{1}{\sqrt{x+1}}\ dx.$$

When I look at this, I see that if I don't replace the $\sqrt{x+1}$ in the denominator with a u, but instead use it as a part of du, this is going to work.

Thus, using $u = \sqrt{x+1}$ and $2\ du = \frac{1}{\sqrt{x+1}}\ dx$ in my substitution, I get

$$\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}}\ dx = \int e^u \cdot 2\ du.$$

This is a *much* simpler integral, with no x terms left in it, so I believe this will indeed be a successful substitution.

$$\begin{aligned} \int e^u \cdot 2\ du &= 2\int e^u\ du \\ &= 2e^u + C \\ &= 2e^{\sqrt{x+1}} + C \end{aligned}$$

5. $\int_2^5 \frac{1}{x\ln(x)}\ dx \qquad (u = \ln(x))$

Let $u = \ln(x)$. Then $\frac{du}{dx} = \frac{1}{x}$, so $du = \frac{1}{x}\ dx$. Looking at the original integral, I see I can rewrite it as

$$\int_2^5 \frac{1}{x\ln(x)}\ dx = \int_2^5 \frac{1}{\ln(x)} \cdot \frac{1}{x}\ dx.$$

Thus I can substitute du in directly for $\frac{1}{x}\ dx$, and u in for $\ln(x)$, and I get

$$\int_2^5 \frac{1}{x\ln(x)}\ dx = \int_{x=2}^{x=5} \frac{1}{u}\ du.$$

Once again, this eliminated all the x terms from the integral and produced something simpler that I know how to antidifferentiate:

$$\begin{aligned} \int_{x=2}^{x=5} \frac{1}{u}\ du &= \ln(u) \text{ from } x=2 \text{ to } x=5 \\ &= \ln(\ln(x)) \text{ from 2 to 5} \\ &= \ln(\ln(5)) - \ln(\ln(2)) \end{aligned}$$