Things to investigate When Graphing a Function

- **Critical Numbers:** Find the critical numbers of f
- **Increasing/Decreasing:** On each interval determined by the critical numbers *and* the points not in the domain of f, find if f is $\uparrow$ or $\downarrow$
- **Local Extrema:** Conclude where there are local mins and max's
- **Potential inflection points:** Find critical numbers of f'
- **Concavity**: On intervals to each side of points where $f''(x) = 0$ or d.n.e., find whether f is $\smile$ or $\frown$. If the concavity changes at a point where f exists, f has an infl pt there: if f' d.n.e, it's an infl pt w/ a vert tangent, if $f'(x) = 0$, then it's an infl pt w/ a hor tangent.

continued...

Things to investigate When Graphing a Function

- **Domain** - are there any points or intervals *not* in the domain of f?
- **Vertical asymptotes** - If a is an isolated point not in the domain (not an interval), is there a vertical asymptote at a, or is it a removable discontinuity? Find $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$
- **Critical Numbers:** Find the critical numbers of f
- **Increasing/Decreasing:** On each interval determined by the critical numbers *and* the points not in the domain of f, find if f is $\uparrow$ or $\downarrow$
- **Local Extrema:** Conclude where there are local mins and max's
- **Potential inflection points:** Find critical numbers of f'
- **Concavity**: On intervals to each side of points where $f''(x) = 0$ or d.n.e., find whether f is $\smile$ or $\frown$. If the concavity changes at a point where f exists, f has an infl pt there: if f' d.n.e, it's an infl pt w/ a vert tangent, if $f'(x) = 0$, then it's an infl pt w/ a hor tangent.

continued...

More things to investigate

- **Horizontal Asymptotes**: Check the limit of $f(x)$ as $x \to \pm\infty$. If either limit is finite, you have a horizontal asymptote on that side. Polynomials and roots do not have horizontal asymptotes.
  A function is most likely to have a horizontal asymptote if it involves an exponential function or is a quotient of two functions.

- **A few key points:** Find the y-values of all the points you've identified as important. Find the y-intercept, and if it's not too painful, find the x-intercepts as well.

In Class Work

Graph the following functions, in either order, without looking at the graph on a calculator. Find all asymptotes, local extrema, and inflection points.

1. $f(x) = \ln(x^2 - 4)$

2. $g(x) = e^{1/x}$

Solution: Graphing $f(x) = \ln(x^2 - 4)$

- **Domain:**

$$x^2 - 4 > 0 \Rightarrow x^2 > 4 \Rightarrow x < -2 \text{ or } x > 2.$$

- **Vertical Asymptotes at $x = \pm 2$?**
  Remembering from the graph of $\ln(x)$ that $\lim_{x\to 0^+} \ln(x) = -\infty$,

$$\begin{aligned} &\text{As} \quad x \to 2^+, \quad x^2 - 4 \to 0^+ \Rightarrow \ln(x^2-4) \to -\infty \\ &\text{As} \quad x \to -2^-, \quad x^2 - 4 \to 0^+ \Rightarrow \ln(x^2-4) \to -\infty \end{aligned}$$

- **Horizontal Asymptotes?**
  As $x \to \infty$, $\ln(x) \to \infty$ (slowly), so $\lim_{x\to\pm\infty} \ln(x^2-4) \to \infty$

Solution: Graphing $f(x) = \ln(x^2 - 4)$

- **Domain:** All $x < -2$ or $x > 2$
- **Vertical Asympotes at $x = \pm 2$:** As $x \to 2^+, -2^-$, $f(x) \to -\infty$
- **Horizontal Asymptotes?** No, as $x \to \pm\infty$, $f(x) \to \infty$

- **Critical Points:** Find where $f'(x) = 0$ or d.n.e. but f does

$$f'(x) = \frac{1}{x^2 - 4} \cdot 2x = \frac{2x}{x^2 - 4}.$$

$$f'(x) = 0 \Leftrightarrow \frac{2x}{x^2 - 4} = 0 \Leftrightarrow x = 0.$$

But $x \notin$ domain of $\ln(x^2 - 4)$, so **f(x) has no local extrema**.

Solution: Graphing $f(x) = \ln(x^2 - 4)$

- **Domain:** All $x < -2$ or $x > 2$
- **Vertical Asymptotes at $x = \pm 2$:** As $x \to 2^+, -2^-$, $f(x) \to -\infty$
- **Horizontal Asymptotes?** No, as $x \to \pm\infty$, $f(x) \to \infty$
- **Critical Points:** f has no local extrema

- **Increasing/Decreasing:** Find where $f'(x)$ positive, negative. Can only change at critical points or points where f d.n.e.
  - To the left of $x = -2$: $f'(-3) = \dfrac{2 \cdot (-3)}{(-3)^2 - 4} < 0$.
    Conclusion: f is decreasing on $(-\infty, -2)$
  - To the right of $x = 2$: $f(3) = \dfrac{2 \cdot 3}{3^2 - 4} > 0$.
    Conclusion: f is increasing on $(2, \infty)$

Solution: Graphing $f(x) = \ln(x^2 - 4)$

- **Domain:** All $x < -2$ or $x > 2$
- **Vertical Asympotes at $x = \pm 2$:** As $x \to 2^+, -2^-$, $f(x) \to -\infty$
- **Horizontal Asymptotes?** No, as $x \to \pm\infty$, $f(x) \to \infty$
- **Critical Points:** f has no local extrema
- **Increasing/Decreasing:** $f \downarrow$ on $(-\infty, -2)$, $f \uparrow$ on $(2, \infty)$
- **Possible Inflection Points:** Where $f''(x) = 0$ or d.n.e. but f does

$$f'(x) = \frac{2x}{x^2 - 4} \Rightarrow f''(x) = \frac{(x^2-4)\cdot 2 - (2x)\cdot(2x)}{(x^2-4)^2} = \frac{-2(x^2+4)}{(x^2-4)^2}$$

There are no values of x at which $f''(x) = 0$. $f''(x)$ only d.n.e. at $x = \pm 2$, which aren't in domain. Thus $f(x)$ **has no inflection points**.

Solution: Graphing $f(x) = \ln(x^2 - 4)$

- **Domain:** All $x < -2$ or $x > 2$
- **Vertical Asympotes at $x = \pm 2$:** As $x \to 2^+, -2^-$, $f(x) \to -\infty$
- **Horizontal Asymptotes?** No, as $x \to \pm\infty$, $f(x) \to \infty$
- **Critical Points:** f has no local extrema
- **Increasing/Decreasing:** $f \downarrow$ on $(-\infty, -2)$, $f \uparrow$ on $(2, \infty)$
- **Possible Inflection Points:** $f(x)$ has no inflection points

- **Concavity:** Find where $f''(x)$ positive, negative. Can only change at possible inflection points or where f or f' d.n.e.
  - To the left of $x = -2$: $f''(-3) = \dfrac{-2\left((-3)^2 + 4\right)}{\left((-3)^2 - 4\right)^2} < 0$

    Conclusion: f is concave down on $(-\infty, 2)$
  - To the right of $x = 2$: $f''(3) = \dfrac{-2(3^2 + 4)}{\left((3)^2 - 4\right)^2} < 0$

    Conclusion: f is concave down on $(2, \infty)$

Solution: Graphing $f(x) = \ln(x^2 - 4)$

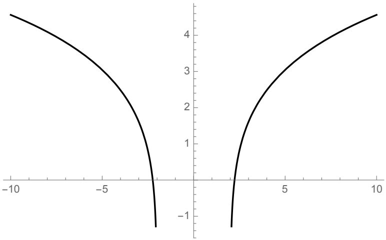
- **Domain:** All $x < -2$ or $x > 2$
- **Vertical Asymptotes at $x = \pm 2$:** As $x \to 2^+, -2^-$, $f(x) \to -\infty$
- **Horizontal Asymptotes?** No, as $x \to \pm\infty$, $f(x) \to \infty$
- **Critical Points:** f has no local extrema
- **Increasing/Decreasing:** $f \downarrow$ on $(-\infty, -2)$, $f \uparrow$ on $(2, \infty)$
- **Possible Inflection Points:** $f(x)$ has no inflection points
- **Concavity:** f is $\frown$ on $(-\infty, 2)$ and on $(2, \infty)$

- **Important points:** We have no inflection points, no local extrema, and no y-intercept, but we should still find a few y-values.
  - x-intercepts:

$$f(x) = 0 \Leftrightarrow \ln(x^2 - 4) = 0 \Leftrightarrow x^2 - 4 = 1 \Leftrightarrow x^2 = 5 \Leftrightarrow x = \pm\sqrt{5} \approx \pm 2.24$$

Solution: Graphing $f(x) = \ln(x^2 - 4)$

- **Domain:**
  All $x < -2$ or $x > 2$
- **Vertical Asympotes:**
  As $x \to 2^+, -2^-$, $f(x) \to -\infty$
- **Horizontal Asymptotes?**
  None. As $x \to \pm\infty$,
  $f(x) \to \infty$
- **Local Extrema:** None
- **Increasing/Decreasing:**
  $f \downarrow$ on $(-\infty, -2)$, $f \uparrow$ on $(2, \infty)$
- **Inflection Points:** None
- **Concavity:**
  f is $\frown$ on $(-\infty, 2)$,$(2, \infty)$
- **Important points:** $(\pm\sqrt{5}, 0)$



Solution: Graphing $g(x) = e^{1/x}$

- **Domain:**

$$\frac{1}{x} \text{ d.n.e. at } x = 0 \Rightarrow \text{Domain: All } x \neq 0$$

- **Vertical Asympotes at $x = 0$?**
  Remembering from the graph of e^x that $\lim_{x \to \infty} e^x = \infty$, $\lim_{x \to -\infty} e^x = 0$,

$$\begin{array}{lll} \text{As} & x \to 0^+, & \frac{1}{x} \to \infty \Rightarrow e^{1/x} \to \infty \\ \text{As} & x \to 0^-, & \frac{1}{x} \to -\infty \Rightarrow e^{1/x} \to 0 \end{array}$$

- **Horizontal Asymptotes?**
  As $x \to \pm\infty$, $\frac{1}{x} \to 0$, so $\lim_{x \to \pm\infty} e^{1/x} \to e^0 = 1$

Solution: Graphing $g(x) = e^{1/x}$

- **Domain:** All $x \neq 0$
- **Vertical Asympotes:** $\lim_{x \to 0^+} g(x) = \infty$, $\lim_{x \to 0^-} g(x) = 0$
- **Horizontal Asymptotes?** As $x \to \pm\infty$, $g(x) \to 1$

- **Critical Points:** Find where $g'(x) = 0$ or d.n.e. but g does

$$g'(x) = e^{1/x} \cdot -\frac{1}{x^2} = -\frac{e^{1/x}}{x^2} \neq 0 \text{ for any } x$$

  Also, only point where $g'(x)$ d.n.e. is $x = 0$, not in domain of g.
  Conclusion: $g(x)$ has no local extrema
- **Increasing/Decreasing:** Find where $g'(x)$ positive, negative. Can only change at critical points or points where g d.n.e.

$$g'(x) = -\frac{e^{1/x}}{x^2} < 0 \text{ for all } x \neq 0$$

  Thus g decreases everywhere on its domain.

Solution: Graphing $g(x) = e^{1/x}$

- **Domain:** All $x \neq 0$
- **Vertical Asymptotes** $\lim_{x\to 0^+} g(x) = \infty$, $\lim_{x\to 0^-} g(x) = 0$
- **Horizontal Asymptotes?** As $x \to \pm\infty$, $g(x) \to 1$
- **Critical Points:** g has no local extrema
- **Increasing/Decreasing:** g decreases for all $x \neq 0$
- **Possible Inflection Points:** Where $g''(x) = 0$ or d.n.e. but g does

$$g'(x) = -\frac{e^{1/x}}{x^2} \Rightarrow g''(x) = -\left(\frac{x^2 \cdot \left(e^{1/x} \cdot -\frac{1}{x^2}\right) - e^{1/x} \cdot 2x}{(x^2)^2}\right)$$

$$= -\frac{-e^{1/x} - 2xe^{1/x}}{x^4} = \frac{(2x+1)e^{1/x}}{x^4}$$

$$\Rightarrow g''(x) = 0 \Leftrightarrow 2x + 1 = 0 \Leftrightarrow x = -\frac{1}{2}$$

Conclusion: Only point where g *might* have inflection point: $x = -\frac{1}{2}$

Solution: Graphing $g(x) = e^{1/x}$

- **Domain:** All $x \neq 0$
- **Vertical Asymptotes** $\lim_{x\to 0^+} g(x) = \infty$, $\lim_{x\to 0^-} g(x) = 0$
- **Horizontal Asymptotes?** As $x \to \pm\infty$, $g(x) \to 1$
- **Critical Points:** g has no local extrema
- **Increasing/Decreasing:** g decreases for all $x \neq 0$
- **Possible Inflection Points:** At $x = -\frac{1}{2}$

- **Concavity:** Find where $g''(x)$ +/-. May only change at $x = -\frac{1}{2}$, 0
  - To the left of $x = -\frac{1}{2}$: $g''(-1) = \frac{e^{1/(-1)}\left(1+2(-1)\right)}{(-1)^4} = \frac{(+)(-)}{(+)} < 0$
    Conclusion: g is concave down on $(-\infty, -\frac{1}{2})$
  - Btwn $x = -\frac{1}{2}$ and $x = 0$: $g''(-\frac{1}{4}) = \frac{e^{-4}\left(1+2(-\frac{1}{4})\right)}{(-\frac{1}{4})^4} = \frac{(+)(+)}{(+)} > 0$
    Conclusion: g is concave up on $(-\frac{1}{4}, 0)$
  - To the right of $x = 0$: $g''(1) = \frac{e^{1}\left(1+2(1)\right)}{(1)^2} > 0$
    Conclusion: g is concave up on $(1, \infty)$

Solution: Graphing $g(x) = e^{1/x}$

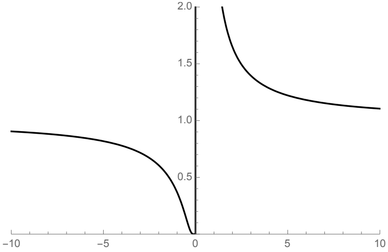
- **Domain:** All $x \neq 0$
- **Vertical Asymptotes** $\lim_{x \to 0^+} g(x) = \infty$, $\lim_{x \to 0^-} g(x) = 0$
- **Horizontal Asymptotes?** As $x \to \pm\infty$, $g(x) \to 1$
- **Critical Points:** g has no local extrema
- **Increasing/Decreasing:** g decreases for all $x \neq 0$
- **Possible Inflection Points:** At $x = -\frac{1}{2}$
- **Concavity:** g is $\frown$ on $(-\infty, -\frac{1}{2})$, $\smile$ on $(-\frac{1}{2}, 0) \cup (0, \infty)$

- **Important points:** We have no local extrema and not exactly a y-intercept
  - Inflection point: $g(-\frac{1}{2}) = e^{-2} \approx 0.14$
  - x-intercepts: $g(x) = 0 \Leftrightarrow e^{1/x} = 0$: Impossible, so no x-intercepts
  - One point to right of $x = 0$: $g(1) = e^1 \approx 2.72$
- **Can g cross $y = 1$?** No, because it's always decreasing. If it crosses it and then becomes asymptotic to it, it would have to change direction.

Solution: Graphing $g(x) = e^{1/x}$

- **Domain:**
  All $x \neq 0$
- **Vertical Asympotes:**
  $\lim_{x \to 0^+} g(x) = \infty, \ \lim_{x \to 0^-} g(x) = 0$
- **Horizontal Asymptotes?**
  As $x \to \pm\infty$, $g(x) \to 1$
- **Local Extrema:** None
- **Increasing/Decreasing:**
  g decreases for all $x \neq 0$
- **Inflection Points:** $x = -\frac{1}{2}$
- **Concavity:**
  $(-\infty, -\frac{1}{2})$: $\frown$;
  $(-\frac{1}{2}, 0) \cup (0, \infty)$: $\smile$
- **Important points:**
  $(-\frac{1}{2}, 0.14)$, $(1, 2.72)$

On $[-10, 10]$:



On $[-5, 5]$: