Consider the **differential equation**

$$x'' + 16x = f(t),\ x(0) = 0,\ x'(0) = 0$$

where

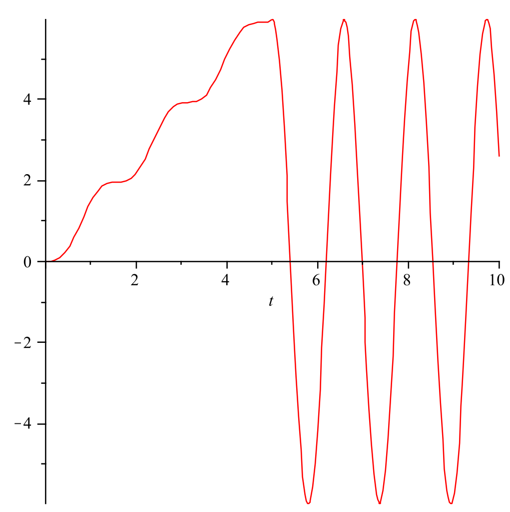
$$f(t) = \begin{cases} 20t, & 0 \le t < 5 \\ 0, & t \ge 5 \end{cases}$$

Transform it (using methods we'll learn) to an **algebraic equation** in a temporary variable s:

$$s^2X(s) + 16X(s) = \frac{20}{s^2} - \left[\frac{20}{s^2} + \frac{100}{s}\right]e^{-5s}$$

After solving for $X(s)$ and writing it in ways that we'll learn are useful, transform it back to the solution to the differential equation,

$$x(t) = \begin{cases} \frac{5}{4}t - \frac{5}{16}\sin(4t) & 0 \le t < 5 \\ -\frac{5}{16}\sin(4t) + \frac{5}{16}\sin(4t-5) & \\ \qquad + \frac{25}{4}\cos(4(t-5)) & t \ge 5 \end{cases}$$



Plan:

1. Learn the basics of what we're doing when we do this transform, called a *Laplace transform*
2. Learn some algebra that we're going to need
3. Learn how undo the transform, i.e. learn *the inverse transform*
4. Learn how to transform a derivative
5. Apply it to some differential equations

Repeat!

Find the Laplace transform $\mathcal{L}\{f(t)\}$ of the following functions. (Feel free to use Maple for the actual antidifferentiation, but evaluate at the limits of integration yourself, for practice with improper integrals.)

1. $f(t) = \begin{cases} 0, & \leq t < 4 \\ t^3, & t \geq 4 \end{cases}$
2. $f(t) = e^{at}$
3. $f(t) = \cos(kt)$
4. $f(t) = \alpha g(t) + \beta h(t)$, where α and β are constants. Find your answer in terms of $G(s) = \mathcal{L}\{g(t)\}$ and $H(s) = \mathcal{L}\{h(t)\}$.
5. Find $\mathcal{L}\{f'(t)\}$, if f is continuous for $t > 0$. Find your answer in terms of $F(s) = \mathcal{L}\{f(t)\}$.

1. $f(t) = \begin{cases} 0, & \leq t < 4 \\ t^3, & t \geq 4 \end{cases}$

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \int_4^\infty t^3 e^{-st}\, dt \\ &= -\frac{6 + 6st + 3t^2s^2 + t^3s^3)e^{-st}}{s^4}\Bigg|_4^\infty \\ &= 0 + \frac{(6 + 24s + 48s^2 + 64s^3)e^{-4s}}{s^4} \text{ if } s > 0 \end{aligned}$$

2. $f(t) = e^{at}$

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \int_0^\infty e^{at} e^{-st}\, dt \\ &= \int_0^\infty e^{-(s-a)t}\, dt \\ &= -\frac{e^{-(s-a)t}}{s-a}\Bigg|_0^\infty \\ &= 0 + \frac{1}{s-a} \text{ if } s > a \\ &= \frac{1}{s-a} \text{ if } s > a \end{aligned}$$

3. $f(t) = \cos(kt)$

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty \cos(kt)e^{-st}\, dt \\
&= \left.\frac{ke^{-st}\sin(kt) - se^{-st}\cos(kt)}{s^2+k^2}\right|_0^\infty \\
&= 0 - \frac{-s}{s^2+k^2} \text{ if } s > 0 \\
&= \frac{s}{s^2+k^2} \text{ if } s > 0
\end{aligned}$$

4. $f(t) = \alpha g(t) + \beta h(t)$, where α and β are constants. Find your answer in terms of $G(s) = \mathcal{L}\{g(t)\}$ and $H(s) = \mathcal{L}\{h(t)\}$.

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty (\alpha g(t) + \beta h(t))e^{-st}\, dt \\
&= \alpha \int_0^\infty g(t)e^{-st}\, dt + \beta \int_0^\infty h(t)e^{-st}\, dt \\
&= \alpha G(s) + \beta H(s)
\end{aligned}$$

5. $\mathcal{L}\{f'(t)\}$

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}\,dt$$

Integration by parts:
Let $u = e^{-st}$, $dv = f'(t)\,dt$
Then $du = -se^{-st}\,dt$, $v = f(t)$

$$\begin{aligned}
\mathcal{L}\{f'(t)\} &= f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt \\
&= (0 - f(0)) + s\mathcal{L}\{f(t)\} \\
&\qquad \text{if } f(t) \text{ is dominated by } e^{st} \\
&= sF(s) - f(0)
\end{aligned}$$

Find the Laplace transform of the DE

$$y'' + 5y' + 4y = e^{-3t}, y(0) = 1, y'(0) = 2.$$

Transforms we already know:

$f(t)$	$\mathcal{L}\{f(t)\}$
t	$\frac{1}{s^2}$
t^3	$\frac{3!}{s^4}$
e^{at}	$\frac{1}{s-a}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0)$ $-s^{n-2} f'(0) - \ldots - s f^{(n-2)}(0)$ $-f^{(n-1)}(0)$

$$\mathcal{L}\{y'' + 5y' + 4y\} = \mathcal{L}\{e^{-3t}\}$$

$$\begin{aligned}
\mathcal{L}\{y''\} &= s^2Y(s) - sy(0) - y'(0) \\
&= s^2Y(s) - s \cdot 1 - 2 \\
\mathcal{L}\{y'\} &= sY(s) - y(0) \\
&= sY(s) - 1 \\
\mathcal{L}\{y\} &= Y(s) \\
\mathcal{L}\{e^{-3t}\} &= \frac{1}{s+3}
\end{aligned}$$

$$s^2Y(s) - s - 2 + 5(sY(s) - 1) + 4Y(s) = \frac{1}{s+3}$$