Use Laplace transforms to solve the IVP

$$y'' + 5y' + 4y = e^{-3t},$$

with

$$y(0) = 1, y'(0) = 2.$$

$$\begin{aligned}
\mathcal{L}\left\{y'' + 5y' + 4y\right\} &= \mathcal{L}\left\{e^{-3t}\right\} \\
\Rightarrow \left(s^2 Y(s) - s - 2\right) & \\
+5\left(sY(s) - 1\right) + 4Y(s) &= \frac{1}{s+3} \\
\Rightarrow (s^2 + 5s + 4)Y(s) &= s + 7 + \frac{1}{s+3}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow Y(s) &= \frac{(s+7)(s+3)+1}{(s+3)(s+1)(s+4)} \\
\Rightarrow \mathcal{L}^{-1}\left\{Y(s)\right\} &= \mathcal{L}^{-1}\left\{\frac{s^2 + 10s + 22}{(s+3)(s+1)(s+4)}\right\} \\
\Rightarrow y(t) &= \text{(by partial fractions)} \\
&= \mathcal{L}^{-1}\left\{\frac{13}{6(s+1)} - \frac{1}{2(s+3)}\right. \\
&\qquad \left. - \frac{2}{3(s+4)}\right\} \\
\Rightarrow &= \frac{13}{6}e^{-t} - \frac{1}{2}e^{-3t} - \frac{2}{3}e^{-4t}
\end{aligned}$$

- A **rational function** is a ratio of polynomials. A rational function $\frac{p(x)}{q(x)}$ is **proper** if $p(x)$ has lower degree than $q(x)$.

- Any proper rational function can be written as the sum of **partial fractions** of just two basic types:

$$\frac{A}{(ax+b)^n} \qquad \text{or} \qquad \frac{Ax+B}{(ax^2+bx+c)^n}.$$

Transforms of Some Basic Functions

| $f(t)$ | $F(s) = \mathcal{L}\{f(t)\}$ |
|---|---|
| $1$ | $\frac{1}{s}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ |
| $e^{at}$ | $\frac{1}{s-a}$ |
| $\sin(kt)$ | $\frac{k}{s^2+k^2}$ |
| $\cos(kt)$ | $\frac{s}{s^2+k^2}$ |
| $\sinh(kt)$ | $\frac{k}{s^2-k^2}$ |
| $\cosh(kt)$ | $\frac{s}{s^2-k^2}$ |

**Also, remember:** Laplace transforms, and therefore inverse Laplace transforms, are **linear transforms**:

$$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\}.$$

Solve the IVP

$$x'' + 16x = f(t), x(0) = 0, x'(0) = 0$$

$$f(t) = \begin{cases} 20t, & 0 \leq t < 5 \\ 0, & t \geq 5 \end{cases}$$

(This can be interpreted as describing the motion of a spring/mass system with an external force acting on the system for only the first 5 seconds; $x(t)$ would represent the displacement of the mass.)

Use Laplace transforms to solve the IVPs below:

1. $y' + 2y = f(t)$, $y(0) = 0$, where $f(t) = \begin{cases} t, & 0 \leq t < 3 \\ 0, & t \geq 3 \end{cases}$

2. $y'' + 4y = f(t)$, $y(0) = 1$, $y'(0) = 0$, where
   $f(t) = \begin{cases} 0, & 0 \leq t \leq 2\pi \\ \sin(t), & t \geq 2\pi \end{cases}$

   **Hint:** You may find it helpful to remember that $\sin(t) = \sin(t - 2\pi)$.