Use Laplace transforms to solve the IVPs below:

1. $y' + 2y = f(t)$, $y(0) = 0$, where $f(t) = \begin{cases} t, & 0 \le t < 3 \\ 0, & t \ge 3 \end{cases}$

   (i) Take the Laplace transform of each side of the DE
   The easiest approach is to first write $f(t)$ in terms of a unit step function.

$$\begin{aligned} f(t) &= \begin{cases} t - 0, & 0 \le t < 3 \\ t - t, & t \ge 3 \end{cases} \\ &= t - t\,\mathcal{U}(t-3) \end{aligned}$$

   Thus

$$\begin{aligned} \mathcal{L}\{y' + 2y\} &= \mathcal{L}\{t - t\,\mathcal{U}(t-3)\} \\ sY(s) - y(0) + 2Y(s) &= \frac{1}{s^2} - \mathcal{L}\{t\,\mathcal{U}(t-3)\} \\ (s+2)Y(s) &= \frac{1}{s^2} - e^{-3t}\mathcal{L}\{t+3\} \\ (s+2)Y(s) &= \frac{1}{s^2} - e^{-3t}\left(\frac{1}{s^2} + \frac{3}{s}\right) \end{aligned}$$

   (ii) Solve for $Y(s)$ and write in a form we can transform back

$$\begin{aligned} Y(s) &= \frac{1}{s^2(s+2)} - e^{-3t}\left(\frac{1}{s^2(s+2)} + \frac{3}{s(s+2)}\right) \\ &= \text{(use convert(, parfrac); on the two sets of rational functions)} \\ &= -\frac{1}{4s} + \frac{1}{2s^2} + \frac{1}{4(s+2)} - e^{-3t}\left(\frac{5}{4s} + \frac{1}{2s^2} - \frac{5}{4(s+2)}\right) \end{aligned}$$

(iii) Take the inverse Laplace transform to solve the IVP

$$\begin{aligned}
\mathcal{L}^{-1}\{Y(s)\} &= \mathcal{L}^{-1}\left\{-\frac{1}{4s}+\frac{1}{2s^2}+\frac{1}{4(s+2)}\right.\\
&\qquad \left. -e^{-3t}\left(\frac{5}{4s}+\frac{1}{2s^2}-\frac{5}{4(s+2)}\right)\right\}\\
y(t) &= -\frac{1}{4}+\frac{1}{2}t+\frac{1}{4}e^{-2t}-\mathcal{L}^{-1}\left\{e^{-3t}\left(\frac{5}{4s}+\frac{1}{2s^2}-\frac{5}{4(s+2)}\right)\right\}\\
&= -\frac{1}{4}+\frac{1}{2}t+\frac{1}{4}e^{-2t}\\
&\qquad -\mathcal{U}(t-3)\left(\frac{5}{4}\cdot 1\Big|_{t\to t-3}+\frac{1}{2}t\Big|_{t\to t-3}-\frac{5}{4}e^{-2t}\Big|_{t\to t-3}\right)\\
&= -\frac{1}{4}+\frac{1}{2}t+\frac{1}{4}e^{-2t}-\mathcal{U}(t-3)\left(\frac{5}{4}+\frac{1}{2}(t-3)-\frac{5}{4}e^{-2(t-3)}\right)\\
&= \begin{cases} -\frac{1}{4}+\frac{1}{2}t+\frac{1}{4}e^{-2t}, & 0\le t<3\\ -\frac{1}{4}+\frac{1}{2}t+\frac{1}{4}e^{-2t}-\frac{5}{4}-\frac{1}{2}(t-3)+\frac{5}{4}e^{-2(t-3)}, & t\ge 3\end{cases}
\end{aligned}$$

2. $y''+4y=f(t)$, $y(0)=1$, $y'(0)=0$, where $f(t)=\begin{cases}0, & 0\le t\le 2\pi\\ \sin(t), & t\ge 2\pi\end{cases}$

Hint: You may find it helpful to remember that $\sin(t)=\sin(t-2\pi)$.

(i) Take the Laplace transform of both sides of the DE.
Once again, the easiest way to get started is to write $f(t)$ in terms of the unit step function.

$$f(t)=\sin(t)\mathcal{U}(t-2\pi).$$

If I in fact write this as $f(t)=\sin(t-2\pi)\mathcal{U}(t-2\pi)$, then I can use the first form of the 2nd translation theorem, which involves slightly fewer steps.

$$\begin{aligned}
\mathcal{L}\{y''+4y\} &= \mathcal{L}\{\sin(t-2\pi)\mathcal{U}(t-2\pi)\}\\
s^2Y(s)-sy(0)-y'(0)+4Y(s) &= e^{-2\pi s}\mathcal{L}\{\sin(t)\}\\
(s^2+4)Y(s)-s &= e^{-2\pi s}\frac{1}{s^2+1}
\end{aligned}$$

(ii) Solve for $Y(s)$ and write in a form we can transform back.

$$\begin{aligned} Y(s) &= \frac{s}{s^2+4} + e^{-2\pi s}\frac{1}{(s^2+1)(s^2+4)} \\ &= \text{(decompose into partial fractions)} \\ &= \frac{s}{s^2+4} + e^{-2\pi s}\left(\frac{1}{3}\cdot\frac{1}{s^2+1} - \frac{1}{3}\frac{1}{s^2+2^2}\right) \end{aligned}$$

(iii) Take the inverse Laplace transform to solve the IVP

$$\begin{aligned} \mathcal{L}^{-1}\left\{Y(s)\right\} &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+4} + e^{-2\pi s}\left(\frac{1}{3}\cdot\frac{1}{s^2+1} - \frac{1}{3}\frac{1}{s^2+2^2}\right)\right\} \\ y(t) &= \cos(2t) + \mathcal{L}^{-1}\left\{e^{-2\pi s}\left(\frac{1}{3}\cdot\frac{1}{s^2+1} - \frac{1}{3}\frac{1}{s^2+2^2}\right)\right\} \\ &= \cos(2t) + \mathcal{U}(t-2\pi)\left(\frac{1}{3}\sin(t)\Big|_{t\to t-2\pi} - \frac{1}{3}\cdot\frac{1}{2}\sin(2t)\Big|_{t\to t-2\pi}\right) \\ &= \cos(2t) + \mathcal{U}(t-2\pi)\left(\frac{1}{3}\sin(t-2\pi) - \frac{1}{6}\sin(2(t-2\pi))\right) \\ &= \cos(2t) + \mathcal{U}(t-2\pi)\left(\frac{1}{3}\sin(t) - \frac{1}{6}\sin(2t)\right) \\ &= \begin{cases} \cos(2t), & 0 \le t < 2\pi \\ \cos(2t) + \frac{1}{3}\sin(t) - \frac{1}{6}\sin(2t), & t \ge 2\pi \end{cases} \end{aligned}$$