Taylor's Theorem:

Let $f(x)$ be a function which is repeatedly differentiable on an interval I containing x_0. Suppose $P_n(x)$ is the n-th order Taylor polynomial based at x_0. Further suppose K_{n+1} is a bound for $|f^{(n+1)}(x)|$ on I. That is,

$$|f^{(n+1)}(x)| \leq K_{n+1} \text{ for all } x \in I$$

Then for all $x \in I$,

$$|f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!}|x - x_0|^{n+1}$$

Let $f(x) = \cos(x)$ and let $x_0 = \frac{\pi}{2}$. You now know how to show that

$$P_5(x) = -(x - \pi/2) + (x - \pi/2)^3/! - (x - \pi/2)^5/5!.$$

1. Use $P_5(2)$ to approximate $\cos(2)$
2. Use Taylor's Theorem to get a feel for how accurate your approximation is.
3. Find a value of n so that $P_n(2)$ approximates $\cos(2)$ accurate within 10^{-10}.

Approximating π

Let $f(x) = \arctan(x)$ and let $x_0 = 0$.

1. Find the exact value of arctan(1).
2. Find $P_5(x)$ for $f(x) = \arctan(x)$.

   (Use Maple to calculate the derivatives of f. Remember, you calculate the second derivative, for instance, by typing in `diff(arctan(x),x,x)`)
3. Use P_5 to approximate arctan(1), and use Theorem 2 to determine how close your approximation is.
4. Use your answers to the above questions to find an approximation for π.
5. If you have time, find a general form for $P_n(x)$, and then use $P_{50}(1)$ to approximate π.