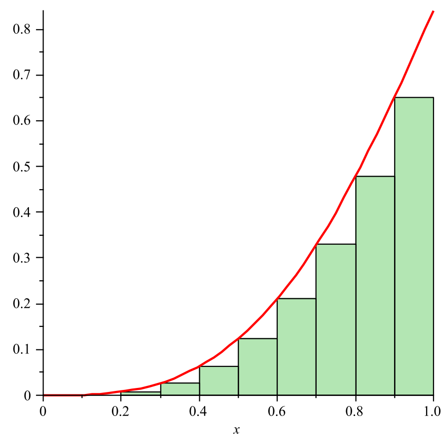
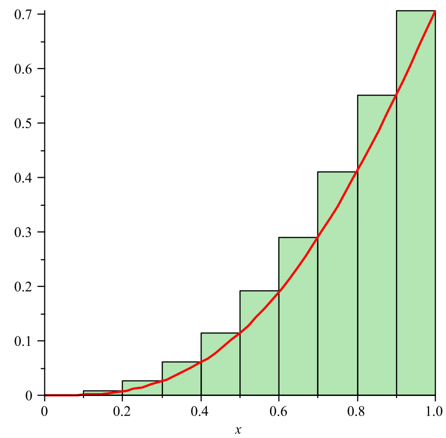
Let $I = \int_0^1 x \sin(x^2)\, dx$

1. Use the leftbox and rightbox commands in Maple to look at L_{10} and R_{10}.



L_{10}



R_{10}

2. Write L_{10} and L_{50} using sigma notation (without using Maple).

When we have the interval $[0, 1]$ and 10 subintervals, the base of all the rectangles will be

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{10} = \frac{1}{10}.$$

Thus the partition is

$$0 < \frac{1}{10} < \frac{2}{10} < \frac{3}{10} < \ldots < \frac{9}{10} < 1.$$

For a left sum, the heights are determined by the left endpoints of each subinterval, which are the first 10 of the 11 numbers in the above partition.

In other words, the heights of the rectangles are $f\left(\frac{i}{10}\right)$ for $i = 0 \ldots 9$. Since the area of a rectangle is just base $\times$ height, and since we're adding up the area of ten rectangles, we therefore have that

$$L_{10} = \frac{1}{10}\left(f(0) + f\left(\frac{1}{10}\right) + f\left(\frac{2}{10}\right) + \ldots + f\left(\frac{9}{10}\right)\right).$$

To put this into sigma notation, we just need to look at the pattern and see where we have consecutive numbers appearing, as opposed to what we have staying identical.

$$\begin{aligned} L_{10} &= \frac{1}{10}\sum_{i=0}^{9} f\left(\frac{i}{10}\right) \\ &= \frac{1}{10}\sum_{i=0}^{9} \frac{i}{10}\sin\left(\frac{i^2}{100}\right) \end{aligned}$$

As for L_{50}, what changes? The width of the subintervals, and therefore the endpoints are the main changes (the heights of the rectangles change also, but that comes for free with changing the endpoints.

$$\Delta x = \frac{1}{50} \qquad 0 < \frac{1}{50} < \frac{2}{50} < \frac{3}{50} < \ldots < \frac{49}{50} < 1.$$

Thus the heights of the rectangles are $f\left(\frac{i}{50}\right)$ for $i = 0 \ldots 49$, and

$$\begin{aligned} L_{50} &= \frac{1}{50}\left(f(0) + f\left(\frac{1}{50}\right) + f\left(\frac{2}{50}\right) + \ldots + f\left(\frac{49}{50}\right)\right) \\ &= \frac{1}{50}\sum_{i=0}^{49} f\left(\frac{i}{50}\right) \\ &= \frac{1}{50}\sum_{i=0}^{49} \frac{i}{50}\sin\left(\frac{i^2}{2500}\right) \end{aligned}$$

3. Write R_{10} and R_{50} using Sigma notation (again, without using Maple).

   We found that for L_{10}, the partition is

   $$0 < \frac{1}{10} < \frac{2}{10} < \frac{3}{10} < \ldots < \frac{9}{10} < 1.$$

   We also know that the only difference between left and right sums are which choice of points in the partition we use. For the left sum, we used the first 10 of these 11 points as the endpoints we plug into f: $\frac{i}{10}$ where i goes from 0 to 9.

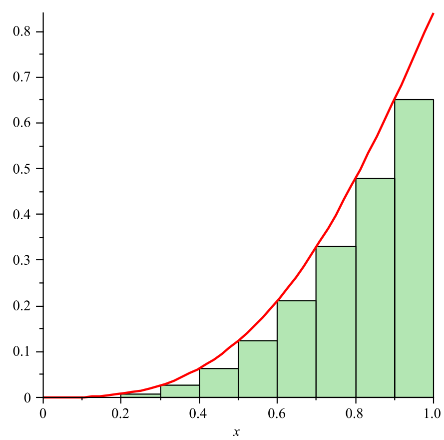
   For a right sum, we use the last 10 of these same 11 points: $\frac{i}{10}$ where i goes from 1 to 10.

The length of the bases of all our rectangles is the same as in L_{10}, and the heights come from the same function.

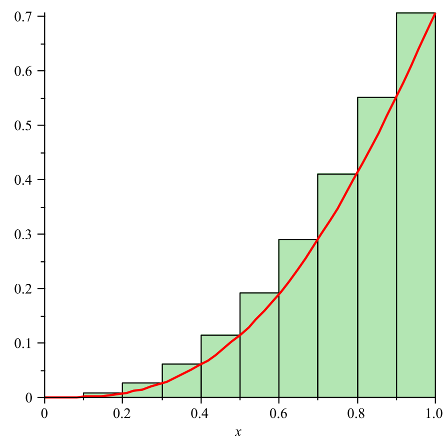
Therefore $R_{10} = \dfrac{1}{10}\displaystyle\sum_{i=1}^{10} \frac{i}{10}\sin\left(\frac{i^2}{100}\right)$.

Notice: If you compare this to L_{10}, all that changed was which values i ranges over.

If you look again at the graphs of L_{10} and R_{10}, this coincides to the fact that the last 9 rectangles in L_{10} are the same in area (just shifted left) as the first 9 rectangles in R_{10}.



L_{10}



R_{10}

Similarly, $R_{50} = \dfrac{1}{50}\displaystyle\sum_{i=1}^{50} \frac{i}{50}\sin\left(\frac{i^2}{2500}\right)$.

4. Without calculating any of them, rank I, L_{10} and R_{10} in increasing order.

   From the pictures I've drawn, I can see that because $x\sin(x^2)$ is increasing, L_{10} under-estimates I while R_{10} over-estimates I. Therefore we have that

$$L_{10} \leq I \leq R_{10}.$$

5. Can you draw any conclusions about how well L_{10} approximates I?

   How good an approximation L_{10} is depends on how close L_{10} is to I. In a real-life application of these approximations, you won't be able to find I exactly, and so you won't be able to tell *exactly* how close the approximation is to I.

So without finding I, what *can* we say about how close L_{10} is to I?

Well, because I is between L_{10} and R_{10}, I know that L_{10} is closer to I than it is to R_{10} ... or at least, it's no farther away from I than it is from R_{10}.

Therefore, even if I can't find the exact value of I, I can at least know that the error in using L_{10} to approximate I, which is $|I - L_{10}|$ is no worse than the difference between L_{10} and R_{10}. That is,

$$\text{error} = |I - L_{10}| \leq |R_{10} - L_{10}|.$$

6. Use the formal definition of the integral to write $I = \displaystyle\int_0^1 x\sin(x^2)\,dx$ as a limit.

   Using the right sum,

$$\int_0^1 x\sin(x^2)\,dx \stackrel{def}{=} \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{i}{n}\sin\left(\frac{i^2}{n^2}\right).$$